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SMA 300 Corrected Module

Statistics & Programming (Kenyatta University)

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KENYATTA UNIVERSITY

DIGITAL SCHOOL OF VIRTUAL AND OPEN LEARNING

IN COLLABORATION WITH

SCHOOL OF PURE & APPLIED SCIENCES
DEPARTMENT: MATHEMATICS AND ACTUARIAL SCIENCE

SMA 300: REAL ANALYSIS I

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VETTED BY: Dr. FREDRICK OLUM

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INTRODUCTION

Welcome to this module. The module is intended to introduce you to foundations of mathematical analysis. By 'mathematical analysis' we mean the study of the real number system, and the mathematical objects that can be constructed from real numbers.

This is an interactive instructional module that uses both action and collaborative learning styles that provide you with diverse online learning experiences and effective learning processes.

In this module, we shall look at the real number system, bounded sets, open sets, closed sets, functions, sequences and series.

We hope that you will find this module exciting, educative, and engaging.

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REAL ANALYSIS I FLOW CHART

WEEK	**TOPIC**
WEEK 1 & 2	THE REAL NUMBER SYSTEM
WEEK 3 & 4	TOPOLOGY OF REAL NUMBERS
WEEK 5 & 6	SEQUENCES
WEEK 7, 8 & 9	SERIES
WEEK 10	COUNTABLE AND UNCOUNTABLE SETS
WEEK 11 & 12	LIMITS AND CONTINUITY OF FUNCTIONS
WEEK 13 & 14	EXAMINATION

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OVERVIEW OF THE COURSE

Week 0: Introduction (Your Context, Your Goals)

This lesson is intended to help you acclimatize to blended learning and to create a community of learners who will motivate each other during the course. You will be required to introduce yourself to your lecturer and colleagues either physically during a face to face session or even online before other academic interactions start.

Week 1 & 2: The Real Number System

In this first lesson, you will be introduced to the real number system $\mathbb{R}$ and some of its properties. In particular the field and order structure of $\mathbb{R}$, rational and irrational numbers and bounded sets will be discussed.

Week 3 &4: Topology of Real numbers

This lesson deals with topology of the real number system. The concepts of interior points, open sets, limit points and closed sets will be discussed

Week 5 & 6: Sequences

In this lesson we will study sequences of real numbers. The main focus of this lesson is properties of convergent and Cauchy sequences.

Week 7, 8 & 9: Series

This lesson deals with infinite series of real numbers. You will be introduced to various tests of convergence of a series; alternating series and absolute and conditional convergence.

Week 10: Countable and Uncountable Sets

In this lesson you will be introduced to countable and uncountable sets. In particular, it will be shown that the set of all integers $\mathbb{Z}$ is countable while $\mathbb{R}$ is uncountable.

Week 11 & 12: Limits and Continuity of Functions

This last lesson deals with limits and continuity of functions. The main item of discussion is $\varepsilon - \delta$ definition limit, pointwise and uniform continuity of a function.

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Week 13 &14: Examination

These two weeks bring together the work you have been doing to an end. This course unit will be examined and will partially contribute to the award of the degree in the programme that you are undertaking.

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PURPOSE OF THE MODULE

The purpose of this module is to enable learners to develop a deeper and more rigorous understanding of the real number system including defining terms and proving theorems about open and closed sets, sequences, series, limits and continuity of functions.

MODULE LEARNING OUTCOMES

By the end of this module the learner should be able to:

i) Construct the real number system and apply the concept of completeness in the set of real numbers and its topology.
ii) Prove and apply theorems associated with sequences of real numbers.
iii) Classify a given series of real numbers as convergent or divergent.
iv) Discuss properties of countable and uncountable sets.
v) Distinguish between point wise and uniform continuity of real-values functions.

COURSE CONTENT

The Real number System: field and order structure, principle of induction, Rational & Irrational numbers, Bounded sets, Supremum and Infimum, Completeness in the set of real numbers. Topology of the Real Numbers: Open sets, limit points, closed sets and closure of sets. Sequences: Limit point of a sequence. Limit superior and Limit Inferior, Convergent sequences, Monotonic sequence, Cauchy sequence, principle of convergent sequences. Series: Partial sum of series, Convergence of series, Absolute and conditional convergence of series. Tests of convergence: Comparison tests, Cauchy's Root test, D'Alembert's Ratio test, Integral test, Alternating series test. Countable and uncountable sets; Functions: Limits, Continuity and Uniform Continuity.

(**Pre-requisite SMA 200: Calculus II)**

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COURSE REQUIREMENTS

This is a blended learning course that will utilize the flex model. This means that learning materials and instructions will be given online and the lessons will be self-guided with the lecturer being available briefly for face to face sessions and support and also on-site (online) most of the time. You are advised to follow the topic flow-chart given so that you cover at least a lesson every week.

You will be required to participate and interact online with your peers and the e-moderator who in this case is your lecturer. Guidelines for the online activities (which we shall keep referring to as e-tivities) will be provided whenever there is an e-tivity. Please note that since the online e-tivities are part of the learning process, they may be graded at the discretion of your e-moderator. Such grading will however be communicated in the e-tivity guidelines and feedback given as soon as possible after the e-tivity. The e-tivities will include but will not be limited to online assessment quizzes, assignments and discussions. There are also assessment questions that you can attempt at the end of every lesson to test your understanding of the lesson. The answers to all the assessment questions are at the end of the module after lesson 6. All the resources that have been used in this module in form of books are available under the resources section after the answers to the questions.

ASSESSMENT

It is important to note that the module has embedded certain learner formative assessment feedback tools that will enable you gauge your own learning progress. The tools include online collaborative discussions forums that focus on team learning and personal mastery and will therefore provide you with peer feedback, lecturer assessment and self-reflection.

I wish you the very best of experiences in this course.

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TABLE OF CONTENTS

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LECTURE ONE

THE REAL NUMBER SYSTEM

1.1 Introduction

In this lesson we introduce the real number system and study some of its properties. We will show that $\Re$ is a complete ordered field and also consider rational and irrational numbers Finally we will introduce the concepts of; bounded sets, infimum and supremum of a subset of $\Re$.

1.2 Learning Outcomes

By the end of this lesson the learner will be able to:

i) Describe $\Re$ as a complete ordered field
ii) Prove that a given number is irrational.
iii) Determine lower bound, upper bound, infimum and supremum of a set
iv) Show that rationals are dense in $\Re$.

1.2.1 Field and Order structure of Real Numbers

1.2.1.1 Notation

We shall denote by:

Z : *the set of all integers*
N : *the set of all positive integers*
Q : *the set of all rational numbers*
Q^{c} : *the set of all irrational numbers*
$\Re$: *the set of all real numbers*
$\forall$: *For all (or for every)*
$\exists$: *There exists (or there is)*
$\exists!$: *There exists a unique element*
$\Rightarrow$: *Implies that*
$\Leftrightarrow$: *If and only if*

1.2.1.2 Algebraic Structure of Real Numbers

Let + denote addition and . denote multiplication. Then

1. For all $x, y \in \Re$, we have
   i) $x + y \in \Re$
   ii) $x.y \in \Re$ (Closure)
2. For all $x, y \in \Re$, we have

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i) $x+y=y+x$

ii) $x.y=y.x$ (Commutative law)

3. For all $x,\ y,\ z\in\Re$, we have

   i) $\left(x+y\right)+z=x+\left(y+z\right)$

   ii) $\left(x.y\right).z=x.\left(y.z\right)$ (Associative law)

4. There exist distinct elements 0 and 1 in $\Re$ such that

   $x+0=x$

   $x.1=x,\quad \forall x\in\Re$

   Number 0 is called the additive identity

   1 is called the multiplicative identity

5. For all $x\in\Re$, the exists a real number *y* such that $x+y=0$. The number $y\in\Re$ is called the additive inverse of *x*.

   Eg if $x=7$ the $y=-7$.

6. For all $x\in\Re$ such that $x\neq 0$, the exists a real number $y\in\Re$ such that $x.y=1$. The number *y* denote by x^{-1} is called the multip[licative inverse of *x*.

   Eg if $x=2$, then $2^{-1}=\frac{1}{2}$.

Definition 1.1: A set with two operations + and . satisfying all the above properties is called a field. Thus the set of all real numbers is a field.

1.2.1.3 Ordering property of real numbers

Definition 1.2: Let F be a field, an order relation $\leq$ on F is a relation such that

i) $\forall\ x,y\in F, x\leq y\ and\ /\ or\ y\leq x$

ii) $if\ \ x<y\ and\ \ y<x\ then\ x=y$

iii) if $x\leq y\ and\ \ y\leq z\ then\ \ x\leq z$

   and if $x\neq y$ we write $x<y\ \ or\ \ y>x$.

A field with an ordered relation satisfying (i), (ii) and (iii) is called an ordered field.

Example 1.1

The set of all real numbers $\Re$ is an ordered field.

E-tivity 1.2.1: Field and Order structure of Real Numbers

Numbering and pacing and sequencing	1.2.1
Title	Field and Order structure of Real Numbers
Purpose	To expose you to the field and order structure of the real number system
Brief summary of overall task	Watch the video on Properties of real numbers by MuchoMath and answer the asked questions.

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Spark	
Individual contribution	• Watch the video on Properties of real number • Discuss the algebraic and order structure of real numbers
Interaction begins	• Post your answers on discussion forum 1.2.1 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take one hour.
Next	Rational and irrational numbers

1.2.2 Rational and Irrational Numbers

1.2.2.1 Rational Numbers

The continuum

We extend the natural numbers by including the number zero and all the negative whole numbers to obtain the set of integers $\{\ldots-2,-1,0,1,2,\ldots\}$

As long as we are in equation of the type $2x-5=0$ we have no whole number solutions, so we need more numbers to include fractions. The next set is called the set of rational numbers (i.e. we extend Z to the set of rational numbers)

Definition 1.3: A rational number is a number of the form $r = {}^{m}\!/_{n}$, such that $(m,n)=1$ and, $m,n \in Z$, $n \neq 0$, that is m and n have no common factor other than 1. The set of rational numbers is denoted by Q , that is $Q=\left\{{}^{m}\!/_{n} : (m,n)=1, m,n \in Z, n \neq 0\right\}$.

The set of rational numbers is closed under both addition and multiplication.

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Lemma 1.1
Between any two rational numbers there exists always a third number.

Proof: let $v_1 \; and \; v_2$ be two rational numbers, then $v_3 = \frac{v_1 + v_2}{2}$ is a rational number between $v_1 \; and \; v_2$.
In fact, this process can be repeated infinitely many times, and so between any two rational numbers, there are infinitely many rationals.

1.2.2.2 Irrational numbers
Definition 1.4: The set of numbers that are not rational cannot be expressed in the form m/n and are called irrational numbers and are denoted by $Q^{\subset}$

Theorem 1.2: There does not exist a rational number r such that $r^2 = 2$, in other words $r = \sqrt{2}$ is an irrational number.
Proof
Suppose to the contrary that $r = \sqrt{2}$ is rational, then

$\sqrt{2} = \frac{m}{n}$ where $(m,n)=1$, $m,n \in Z$.

$\Rightarrow 2 = m^2/n^2 \Rightarrow m^2 = 2n^2$, we see that m^2 is even. This implies that m is also even (because if $m = 2k-1$ is odd then its square $m^2 = 2(2k^2 - 2k)+1$ is also odd). Now let m=2k for $k \in N$, then $4k^2 = 2n^2$, so that $n^2 = 2k^2$. Therefore, n^2 is even, and so n is even.
Then both m and n are even, hence have a common factor 2, which is a contradiction. Hence $\sqrt{2}$ is irrational.

Definition 1.5: The set of all rational and irrational numbers is called the set of real numbers or the continuum denoted by $\Re$. Thus $\Re = Q \cup Q^{\subset}$, the set $\Re$ is closed under addition and multiplication.

1.2.2.4 Intervals
Let a and b be any two real numbers such that $a < b$. Then

a) $(a,b) = \{x : x \in \Re, a < x < b\}$ is called an open interval
b) $[a,b] = \{x \in \Re; a \le x \le b\}$ is called a closed interval
c) $(a,b] = \{x : x \in \Re, a < x \le b\}$ is called half-open/ half-closed interval.
d) $[a,b) = \{x : x \in \Re : a \le x < b\}$ is called half-closed/ half-open interval

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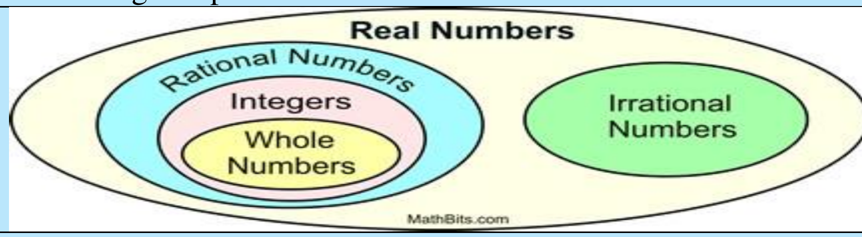
Etivity 1.2.2: Rational and Irrational numbers

Numbering and pacing and sequencing	1.2.2
Title	Rational and Irrational numbers
Purpose	To expose you to the concepts of rational and irrational numbers.
Brief summary of overall task	Watch the videos on Rational and irrational numbers by L Hub and solve the given problems.
Spark	
Individual contribution	• Watch the videos on rational and irrational numbers • Prove that $\sqrt{p}$ is, where irrational p be a prime number • Show that if t is irrational, then $S=\frac{t}{t+1}$ is irrational
Interaction begins	• Post your answers on discussion forum 1.2.2 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take one hour.
Next	Bounded sets

1.2.3 Bounded Sets

Definition 1.6: Let S be a non-empty subset of $\Re$.

a) The set S is said to be bounded above if there exists a number $x \in \Re$ such that $s \leq x$ for all $s \in S$. Such a number x is called an upper bound of S.

b) The set S is said to be bounded below if there exists a number $y \in \Re$ such that $y \leq s$ for all $s \in S$. Such that a number y is called a lower bound of S.

A set S is said to be bounded if it is both bounded above and bounded below. A set is said to be unbounded if it is not bounded.

Example 1.2

i) The set $S = \{x \in \Re : x \leq 4\}$ is bounded above. The set has no lower bound.

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Note that all real numbers greater than 4 are also upper bound of S.

ii) The set $S=\{x \in \Re : x \geq 2\}$ is bounded below. The set has no upper bound. Note that all real numbers less than 2 are also lower bounds of S.

iii) Lastly the set $S=\{x \in \Re : 2 < x \leq 4\}$ isbounded, with lower bound 2 and upper bound 4.

Definition 1.7: Of all the upper bounds of a set S, the least is called a least upper bound or supremum of S denoted by sup S.
That is u is a supremum of S if

i) u is an upper bound of S, and
ii) if v is any upper bound of S, then $u \leq v$
iii) $\forall \in > 0 \exists \ s \in S : u - \in < s < u$.

Definition 1.8 (Bounded set): If S is bounded below, then of all the lower bounds, the greatest is called a greater lower bound, or an infimum of S denoted by inf S. That is u is an infimum of S if:

i) u is a lower of S and
ii) if v is a lower bound of S, then $u \geq v$
iii) $\forall \in > 0 \exists s \in S : u < s < u + \varepsilon$.

If a supremum of a set belongs to the set, then it is called the maximal element of the set. Similarly if an infimum of a set belongs to the set, then it is called the minimum element of the set.

Example 1.3

a) For the set $S=[-2,3]$ the
Supremum = maximal element = 3 and the
Infimum = minimum element = -2

b) For the set $S=(1,3)$, S has neither maximal or minimum element. The supremum of S is 3 while the infimum is 1.

1.2.3.1 Completeness Axiom of the set of real numbers $\Re$

Every non-empty subset $S \ of \ \Re$ which is bounded above has a supremum in $\Re$ and every non-empty subset $S \ of \ \Re$ which is bounded below has an infimum in $\Re$.

The property of $\Re$ as an ordered field taken together with the completeness axiom of $\Re$ makes $\Re$ a complete ordered field.

Example 1.4

The set of rational numbers does not satisfy the completeness axiom. For consider a non-empty subject $S \ of \ Q$, $S=\{x \in Q : \ Q < x^2 < 2\}$. Then S is bounded and sup $S=\sqrt{2}$, which does not belong to $Q. i.e \sqrt{2} \notin Q$.

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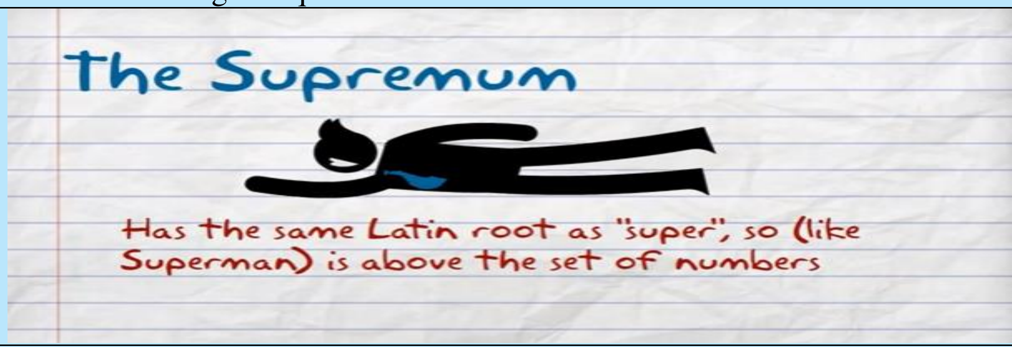
E-tivity 1.2.3: Bounded sets

Numbering and pacing and sequencing	1.2.3
Title	Bounded sets
Purpose	To help you compute infimum, supremum of a given set and determine whether it's bounded or not
Brief summary of overall task	Watch the video on supremum and infimum by Shamim Hussain and give solutions to the given questions.
Spark	
Individual contribution	• Watch the video on supremum and infimum • Find the supremum, infimum, maximal and minimum elements of the following sets: a) $S = \{r \in Q : 0 \le r \le \sqrt{2}\}$ b) $S = \left\{\frac{1}{m} + \frac{1}{n} : n, m \in \mathrm{N}\right\}$ c) $S = \left\{\frac{m}{n} : m, n \in \mathrm{N}, m < n\right\}$ d) $S = \left\{1 + \frac{1}{n} : n \in \mathrm{N}\right\}$ e) $S = \left\{(-1)^n\left(1 + \frac{1}{n}\right) : n \in \mathrm{N}\right\}$ f) $S = \left\{\frac{n}{n+1} : n \in \mathrm{N}\right\}$
Interaction begins	• Post your answers on discussion forum 1.2.3 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute

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	• Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take one hour.
Next	Density of Rational Numbers in $\Re$

1.2.4 Density of Rational Numbers in $\Re$

The set of rational numbers is "dense" in $\Re$, in the sense that given any two real numbers, there is a rational number between them (Infact, there are infinitely many rational numbers).

Proposition 1.3: If $S = \{1/n : n \in N\}$ then inf $S = 0$

Proof-clear

Lemma 1.4: If $t > 0, \exists n_{\in} \in N$ such that $0 < 1/n_{\varepsilon} < t$.

Proof: Since inf $\{1/n : n \in N\} = 0, \text{ and } t > 0$, then t is not a lower bound for the set $\{1/n : n \in N\}$. Thus there exists $n_{\in} \in \aleph$ such that $0 < 1/n_{\varepsilon} < t$.

Theorem 1.5 (The Density): If x and y are any real numbers with $x < y$ then there exists a rational number $r \in Q$ such that $x < r < y$.

Proof

Assume that $x > 0$. Since $y - x > 0$, it follows from Lemma 1.4 that $\exists \; n \in N$ such that $1/n < y - x$. Therefore, we have $nx < ny$. Since $nx > 0$, $\exists \; m \in N$ such that $m - 1 \le nx < m$ (show this). Therefore $m \le nx + 1 < ny, \text{ where } nx < m < ny$. Thus, the rational number $r = m/n$ satisfies $x < r < y$.

Corollary 1.6: If x and y are real numbers with $x < y$, then there exists an irrational number such that $x < z < y$.

Proof

By Theorem 1.5 between the real numbers $\frac{x}{\sqrt{2}} \text{ and } \frac{y}{\sqrt{2}}$, there is a rational number $r \neq 0$ (why?) such that $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$.

Then $z = r\sqrt{2}$ is an irrational number (why?) such that $x < z < y$.

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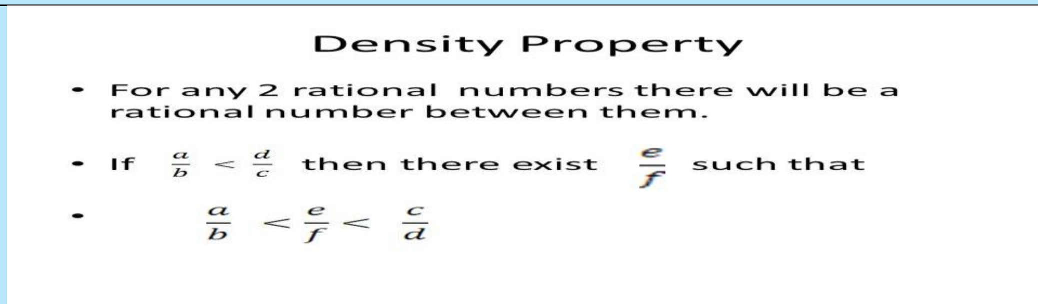
E-tivity 1.2.4: Density of Rational Numbers in $\Re$

Numbering and pacing and sequencing	1.2.4
Title	Density of Rational Numbers in $\Re$
Purpose	To help you show that rationals are dense in $\Re$
Brief summary of overall task	Watch the video on [Between two real numbers there is a rational number]() by Jason Aubrey and answer the asked questions.
Spark	
Individual contribution	• Watch the video between two real numbers there is a rational number Answer the questions; a) If u>0 is any real number and $x < y$, show that there exists a rational number r such that $x < ru < y$. (Hence the set $\{ru : r \in Q\}$ is dense in $\Re$) b) Complete the proof of theorem 1.5 by removing the assumption that $x > 0$
Interaction begins	• Post your answers on discussion forum 1.2.4 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take one hour.
Next	Open sets

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1.3 Assessment Questions

Let A and B be any two bounded nonempty subsets of real numbers and let $A+B=\{a+b : a \in A,\ b \in B\}$. Show that $\sup(A+B)=\sup A+\sup B$.

1.4 References

1. Chidume, C. E. (2006). *Foundations of Mathematical Analysis*. The Abdus Salam ICTP, Triese, Italy.

2. Bartle, R. G. (1976). *The elements of Real Analysis*. John Wiley and Sons, Inc. New York.
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3. Marsden, J. E. (1974). *Elementary Classical Analysis*. W. H. Freeman and Company, San Francisco.

4. Rudin, W. (1966). *Principles of Mathematical Analysis*. McGraw-Hill, Inc. New York.

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LECTURE TWO

TOPOLOGY OF REAL NUMBERS

2.1 Introduction

In this lesson we will study the topology of the real number system $\Re$. We will discuss the concepts of interior points of a set, open sets, limit points of a set and closed sets

2.2 Learning Outcomes

By the end of this lesson the learner will be able to:

i) Discuss basic topological concepts of open sets and prove some results on open sets.
ii) Discuss basic topological concepts of closed sets and prove some results on closed sets

1.2.1 Open sets

Definition 2.1: A subset $N \, of \, \Re$ is said to be a neighbourhood of a point $x \in \Re$ if there exists an open interval $(a,b) = (x - \in, x + \in)$ such that $x \in (a,b) \subset$ N. We call $(a,b) = (x - \in, x + \in)$ an $\in -$neighbourhood of x for some $\in > 0$.

Example 2.1

1. The set $\Re$ is nbd (neighbourhood) of each of its points
2. The set of rational numbers is not a nbd of each of its points (why?)
3. An open interval (a,b) is a nbd of its points.
4. The empty set ϕ is a nbd of each of its points.

Lemma 2.1
If N_1 and N_2 are nbds of a point $x \in \Re$ then $N_1 \cap N_2$ is also a nbd of x.
Proof (Exercise).

Definition 2.2: Let A be a subset of $\Re$. A point $x \in A$ is called an interior point of A if $\exists$ an open interval (a,b) such that $x \in (a,b) \subset A$. The set of all interior points of a set A is called the interior set denoted by A° or int A.

Example 2.2

1. Let $A = [2,4]$, a closed interval; then $7/2$ is an interior point of A, but neither 2 nor 4.
2. Note that $[2,4]^{\circ} = (2,4)$ but $(2,4)^{\circ} = (2,4)$

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3. $N^{\circ} = \phi, \quad Z^{\circ} = \phi, \quad Q^{\circ} = \phi, \Re^{\circ} = \Re$

Definition 2.3: A subset B *of* $\Re$ is open in $\Re$ if for each $x \in B$ there exists a nbd V of x such that $x \in V \subset B.$ That is B is open if it contains an open interval about each of its points. We also see that a set B is open if and only if $B^{\circ} = B$.

Example 2.3

i) Every open interval is an open set.
ii) The closed interval is not open
iii) The empty set is open, since ϕ has no element, so there is no element in ϕ which is not an interior point.
iv) The sets N, Z, Q are not open sets.
v) The set $\Re$ is open since $\Re^{\circ} = \Re$

The following basic result describes the manner in which open sets relate to the operations of the union and intersection of sets in $\Re$.

Theorem 2.2 (Open sets)

a) The union of an arbitrary collection of open subsets in $\Re$ is open.
b) The intersection of any finite collection of open sets in $\Re$ is open.

Proof

a) Let $U = \bigcup_{\alpha \in 1} A_{\alpha}$ where A_{α} is open for each α. If $U = \phi$, then U is open. If $U \neq \phi$, let $x \in U$, then $x \in A_{\alpha}$ for some α. Since A_{α} is open, there exist an open interval (a, b) such that $x \in (a,b) \subset A_{\alpha} \Rightarrow x \in (a,b) \subset U$ (since $A_{\alpha} \subset U \ \forall \ \alpha$). Hence U is a nbd of x implying that U is open, since x was arbitrary.

b) Let $(A)_{i=1}^{n}$ be a finite family of open sets. We show that $B = \bigcap_{i=1}^{n} A_i$ is open.
If $B \neq \phi$, then B is open. Let $x \in B$, then $x \in A_i$ for all $i = 1, \ldots, n.$
Since A_i is open, then $x \in (x - \in_i, x + \in_i) \subset A_i$ for some $\in_i > 0$ for all i.
Letting $\in = min\{\in_i\}, i = 1, 2, \ldots, n.$ Then $x \in (x - \in, x + \in) = U$. So
$x \in U \subset \bigcap A_i = B.$ Hence B is open.

Note that, the intersection of an arbitrary collection of open sets need not be open.
For example:

a) Let $G_n = \left(\frac{-1}{n}, \frac{1}{n} \right), \, for \; n \in N.$

i. e $G_1 = (-1, 1), \; G_2 = \left(\frac{-1}{2}, \frac{1}{2} \right), \; G_3 = \left(\frac{-1}{3}, \frac{1}{3} \right), \ldots$

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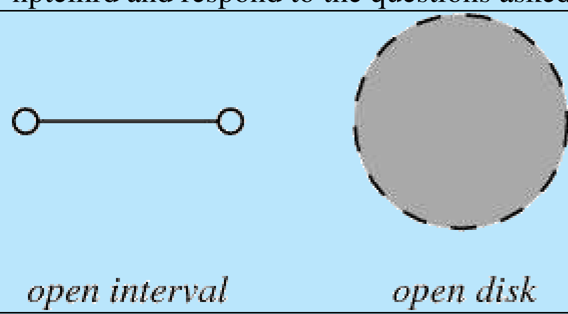
Then clearly, G_n is open for each $n \in \mathbb{N}$. However, the intersections. $G = \bigcap_{n=1}^{\infty} G_n = \{0\}$, which is not open for 0 is not an interior point of $\{0\}$ since $(0-\in, 0+\in) \not\subset \{0\}$ for every $\in > 0$.

b) Let $A_n = \left(2 - \frac{1}{n}, 3 + \frac{1}{n}\right)$ for $n \in \mathbb{N}$.

Then A_n is open for each $n \in \mathbb{N}$. However, $\bigcap_{n=1}^{\infty} A_n = [2,3]$ which is not open.

Etivity 2.2.1: Open sets

Numbering and pacing and sequencing	2.2.1
Title	Open sets
Purpose	To help you determine interior points a set and prove theorems associated with open sets.
Brief summary of overall task	Watch the videos on Topology of real numbers by Mathetic and Open sets by nptelhrd and respond to the questions asked
Spark	
Individual contribution	• Watch the two videos • Find the interior of a) $A = \left\{1 + \frac{1}{10^n} : n \in \mathbb{N}\right\}$ b) $B = \left\{\frac{m}{n} : m, n \in \mathbb{N}, m < n\right\}$ c) $C = \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$ d) $D = \{(n, n+1) : n \in \mathbb{N}\}$ e) $E = (-\infty, -2) \cup (2,3) \cup \{4\} \cup [7, \infty)$ • Reflect on the proof of Theorem 2.2
Interaction begins	• Post your answers on discussion forum 2.2.1 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have

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	posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take one hour.
Next	Closed sets

2.2.2 Closed sets

Definition 2.4 (Limit point): A real number p is a limit point (or an accumulation point or a cluster point) of S if every neighbourhood of p contains at least one point of S different from p. That is if N is the neighbourhood of p, then p is a limit point of S if $\left(N-\{p\}\right)\bigcap S \neq \phi$.

The set of all limit points of S is denoted by $S^{'}$ and is called the derived set of S.

Example 2.4

Let $(0,1) \subset [0,1]$. Then every member of $[0,1]$ is a limit point of $(0,1)$. Thus though $\{0,1\}$ do not belong to $(0,1)$ they are its limit points for the nbd $(1-\in, 1+\in)$ of 1 contains infinitely many elements of $(0,1)$.

Note that from this we see that a limit point may or may not belong to the set.

Example 2.5

a) The sets N, Z *and* ϕ have no limit points.
b) Every real number is a limit point of the sets Q *and* $\Re$.

Lemma 2.3

Finite sets have no limit points.

Proof: Consider the set $A = \{x_1, x_2, ..., x_n\}$. Let p be an arbitrary real number and define $d_1 = |x_1 - p|, ..., d_n = |x_n - p|$.

Let $r = \min\{d_1, ..., d_n\}$. Then the nbd $N_p\left(\frac{r}{2}\right)$ of p contains no point of A. Hence p is not a limit point of A. Since p was arbitrary, A has no limit points.

Next we have a characterization of a limit point of a set.

Theorem 2.4

A real number p is a limit point of a subset S of $\Re$ iff every nbd of p contains infinitely many points of S.

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Proof

($\Rightarrow$) Let p be a limit point of S and Let N_p be a nbd of p We show that N_p has infinitely many points. Suppose to the contrary that N_p has a finite number of points of S distinct from p. Then by the above lemma p is not a limit point of S, which is a contradiction? Hence N_p contains infinitely many points of S. Thus every nbd of p has infinitely many of S, since N_p was arbitrary.

($\Leftarrow$) Conversely, given that every nbd of p has infinitely many points of S we have that every nbd of p contains a point of S different from p. Hence p is a limit point of S.

Definition 2.5 (Closed set): Let X be a set and $S \subset X$. Then S is said to be closed in X if it contains all its limit points. That is, if S is closed and s is a limit point of S then $s \in S$.

Example 2.6

1. The ϕ is closed for there is no element which is not a limit point of ϕ.
2. The set $[a,b]$ is closed since a, b are limits points of A and $a, b \in A$.
3. Every finite set is closed for there is no limit point that does not belong to the set since finite sets have no limit points.

Theorem 2.5

A set A is closed iff its complement $A^{\subset}$ is open.

Proof

Suppose A is closed. We show that $A^{\subset}$ is open. If $A^{\subset} = \phi$, *then* $A^{\subset}$ is open. Suppose that $A^{\subset} \neq \phi$ and let $x \in A^{\subset}, then\, x \notin A$. Since A is closed x is not a limit point of A so there exists a neighbouhood N_x of x such that $N_x \bigcap A = \phi$, which implies that $N \subset A^{\subset}$. Thus $x \in N \subset A^{\subset}$ implying that $A^{\subset}$ is open.

Conversely, suppose that $A^{\subset}$ is open. We want to show that A is closed. Let x be a limit point of A. Then every neighbourhood of x in such that $N \cap A \neq \phi$, thus $x \notin A^{\subset}$ since $A^{\subset}$s open. There $x \in A$ as required. Hence A is closed.

Theorem 2.6

A subset S *of* $\Re$ is closed iff it contains of all of its limit points.

Proof

($\Rightarrow$) Suppose that S is closed and let $x \in S'$, we want to show that $x \in S$. Suppose to the contrary that $x \notin S$, then $x \in S^{\subset}$, which is open. Thus there exists a neighbourhood N_x of x such that $N_x \subset S^{\subset}$, which contradicts the fact that $x \in S'$. Hence $x \in S$.

($\Leftarrow$) Conversely, suppose that $S' \subseteq S$. We want to show that S is closed, equivalently $S^{\subset}$ is open. Let $x \in S^{\subset}$ then $x \notin S'$, so there exists neighbourhood N_x of x such that $N_x \cap S = \phi$, Thus $N_x \subset S^{\subset}$. Hence $S^{\subset}$ is open and so S is closed.

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Theorem 2.7

a) The intersection of an arbitrary collection of closed sets in $\Re$ is closed.
b) The union of any finite collection of closed sets in $\Re$ is closed.

Proof (Exercise)

Note: The finiteness condition in part (b) cannot be removed. For consider the example $A_n = [1/n, 1]$ which is closed but $\bigcup_{n-1}^{\infty} A_n = (0,1]$ which is not closed.

Definition 2.6 (Closure): Let $S \subset \Re$, the set of all limit points of S is denoted by S' and is called the derived set. The closure of S denoted by S is defined to be $\overline{S} = S \cup S'$.
Note that if S is closed then $\overline{S} = S$.

Theorem 2.8

If S is a non-empty closed and bounded subset of $\Re$ then S has a maximum and a minimum element.

Proof

Since S is bounded above, then S has a supremum (by the completeness axiom). Let $m = Sup\, S$. Now given any $\in > 0$, $m - \in$ is not an upper bound for S. If $m \notin S$, there exists an $x \in S$ such that $m - \in < x < m$. But this implies that m is a limit point of S. Since S is closed we must have $m \in S$. Hence $m = \max S$. Similarly it can be shown that inf S= min S (show this as an exercise).

Definition 2.7 (Dense set): A subset A *of* $\Re$ is said to be dense in $\Re$ if $\overline{A} = \Re$. That is every point of X is a limit point of A.

Example 2.7

The set of rational numbers is dense in $\Re$. For let $x \in \Re$, *if* $x \in Q$. we are done, otherwise if $x \notin Q$ then every neighbourhood N_x *of* x contains at least one other rational number. Therefore x is a limit point of Q and so Q is dense in $\Re$, *i.e* $\overline{Q} = \Re$.

E- tivity2.2.2: Closed sets

Numbering and pacing and sequencing	2.2.2
Title	Closed sets
Purpose	To help you determine limit points a set and prove theorems associated with closed sets.

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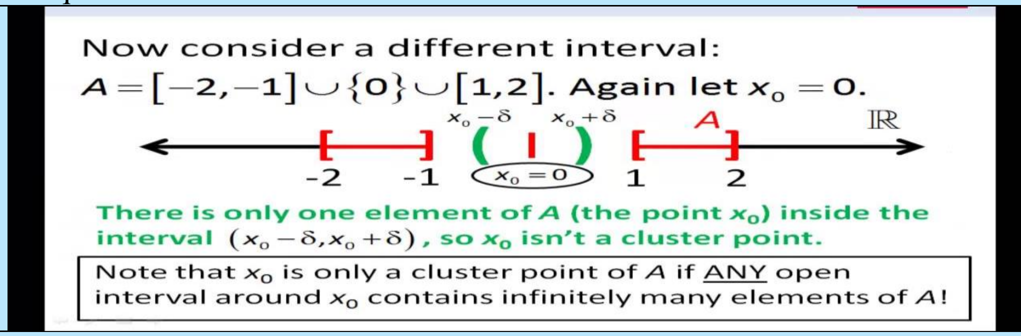
Brief summary of overall task	Watch the videos on Topology of real numbers by Mathetic and Complement, union and intersection of open and closed sets by Nightwing52 and respond to the questions asked.
Spark	
Individual contribution	• Watch the two videos • Find the derived sets of the following sets and state whether they are closed or not: a) $S=\{all\ \text{int}\, egers\}$ b) $S=\cup\{(n,n+1): n\in \mathrm{N}\}$ c) $S=\left\{x: x=\frac{1}{n},\ n\in \mathrm{N}\right\}$ d) $S=\left\{1+\frac{(-1)^n}{n}: n\in \mathrm{N}\right\}$ e) $S=\left\{1,-1,1\frac{1}{2},-1\frac{1}{2},1\frac{1}{3},-1\frac{1}{3},1\frac{1}{4},-1\frac{1}{4},...\right\}$ • Reflect on the proves of Theorems 2.5, 2.6, 2.7 and 2.8
Interaction begins	• Post your answers on discussion forum 2.2.2 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take one hour.
Next	Convergence of sequences

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2.3 Assessment Questions

If A is open and B is closed, prove that $A \backslash B$ is open.

2.4 References

1. Chidume, C. E. (2006). *Foundations of Mathematical Analysis*. The Abdus Salam ICTP, Triese, Italy.

2. Bartle, R. G. (1976). *The elements of Real Analysis*. John Wiley and Sons, Inc. New York.
https://hayanihamudi.files.wordpress.com/2014/01/the-elements-of-real-analysis-by-robert-g-bartle.pdf

3. Marsden, J. E. (1974). *Elementary Classical Analysis*. W. H. Freeman and Company, San Francisco.

4. Rudin, W. (1966). *Principles of Mathematical Analysis*. McGraw-Hill, Inc. New York.

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LECTURE THREE

SEQEUNCES

3.1 Introduction

In this lesson we will study sequences. We will first consider convergence and divergence of sequences and prove some results on convergence of sequences. Finally we will discuss Cauchy sequences and their properties.

3.2 Learning Outcomes

By the end of this lesson the learner will be able to:

i) Investigate convergence of sequences
ii) Prove and apply some results on convergence of sequences
iii) Prove and apply some results on subsequences and Cauchy sequences
iv) Find the limit superior and limit superior of a given sequence.

3.2.1 Convergence of Sequences

Loosely speaking a sequence is a collection of real numbers obeying a particular pattern or definite rule or order.

Definition 3.1: A sequence of real numbers is a function whose domain is the set of natural numbers $\mathrm{N} = \{1, 2, ...\}$ and range is a subset of $\Re$. That is $f : \mathrm{N} \to \Re$ where for any $n \in \mathrm{N}$ there exists a unique number α_n such that $f(n) = \alpha_n$.

We denote a sequence by $\{\alpha_n\}$ or by listing the elements $(\alpha_1, \alpha_2, ..., \alpha_n, ...)$ and call α_n the nth term of the sequence.

Example 3.1

a) $x = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8} ...\right\}$ is the sequence of the even numbers.

b) $s = \left\{1, 1/2, 1/4, 1/8, 1/16, ...\right\}$

$= \left\{1/2^n\right\}$, give the formula of the sequence.

A sequence $\{\alpha_n\}$ is a constant sequence if $\alpha_n = \alpha \;\; \forall\, n.$ A sequence is said to be bounded if there is a real number M such that
$|\alpha_n| \le M \;\forall n$.

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Example 3.2

The sequence $s = \left\{ 1/2^n \right\}$ is bounded for $|\alpha_n| \leq 1$ or $0 \leq \alpha_n \leq 1$.

But the sequence $s = \{2, 4, 6, 8, ...\}$

$= \{2n\}$ is not bounded.

Definition 3.2 (Limit point and convergence of sequences): A sequence $\{\alpha_n\}$ of real numbers is said to converge to $\alpha \; in \; \Re$ if for every $\in > 0$ there exist a number $N = N(\varepsilon)$ such that

$|\alpha_n - \alpha| < \in \alpha$, $\forall n > N$.

The number α is called the limit point of the sequence and we write

$\lim_{n\to\infty} \alpha_n = \alpha \;\; or \;\; \alpha_n \to \alpha \; as \; n \to \infty$.

If the limit of a sequence exists then the sequence is said to be convergent otherwise the sequence in said to be divergent.

Example 3.3

Show that $\lim_{n\to\infty} \left(\frac{1}{n} \right) = 0$.

Proof

We want to show that given $\in > 0$ there is a number $N = N(\in)$ such that

$|\alpha_n - 0| < \in, \forall n > N.$

Now,

$|\alpha_n - 0| = \left| \frac{1}{n} - 0 \right| < \in$

$\frac{1}{n} < \in$

$\Leftrightarrow \quad n > 1/\varepsilon$

Certainly $1/n < \in$, $\forall \; n > 1/\varepsilon$.

Choose N such that $N = 1/\varepsilon$.

Example 3.4

Show that $\lim_{n\to\infty} \alpha_n = 1$

where $\alpha_n = 1 + (-1)^n \frac{1}{n^2}$

Proof

Let $\in > 0$ be given. Then

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$|\alpha_n - 1| < \in \Rightarrow$

$\left|1 + (-1)^n \frac{1}{n^2} - 1\right| < \in$

$\left|\frac{(-1)^n}{n^2}\right| < \in$

$\Rightarrow n^2 > \frac{1}{\varepsilon}$

$n > \frac{1}{\sqrt{\varepsilon}}$

Choose N such that, $N = {1}/{\sqrt{\in}}$ integer then

$|\alpha_n - 1| < \varepsilon, \quad \forall\, n > N$.

Example 3.5

Show that $\lim_{n\to\infty}\left({1}/{n^2+1} \right) = 0$

${1}/{n^2+1} < {1}/{n^2} < {1}/{n} \to 0$

Example 3.6

Show that $\lim_{n\to\infty}\left(\frac{3n+2}{n+1}\right) = 3$.

Proof

Given $\in > 0$ we have

$\left|\frac{3n+2}{n+1} - 3\right| < \in$

$\Leftrightarrow \left|\frac{3n+2-3n-3}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \left|\frac{1}{n+1}\right| < \in$

$\Leftrightarrow \frac{1}{n+1} < \in$

$\Leftrightarrow n+1 > {1}/{\in}$

$\Leftrightarrow n > {1}/{\in} - 1$

Choose N such that $N = {1}/{\varepsilon} - 1$ integer then

$|\alpha_n - 3| < \in, \quad \forall\, n > N$.

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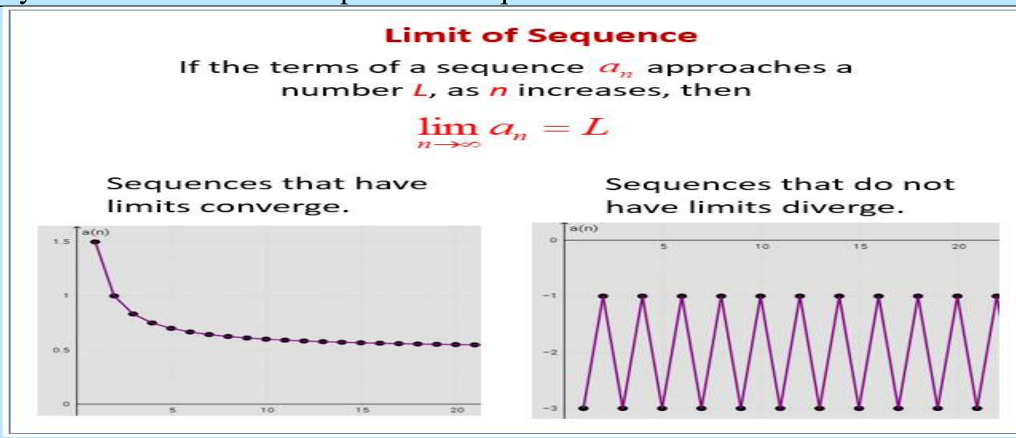
E--tivity 3.2.1: Convergence of sequences

Numbering and pacing and sequencing	3.2.1
Title	Convergence of sequences
Purpose	To help you prove limit of sequences by first principles
Brief summary of overall task	Watch the video on Limit of a sequence by Maurice Koster and Limit problems by Zor Shekhtman and respond to the questions asked
Spark	
Individual contribution	• Watch the two videos • Show that $\lim_{n\to\infty}\left(\frac{n-1}{n}\right)=1$. • Show that $\lim_{n\to\infty}(-1)^{n+1}\left(\frac{n-1}{n}\right)$ does not exist.
Interaction begins	• Post your answers on discussion forum 3.2.1 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take one hour.
Next	Properties of limits

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3.2.2 Properties of sequences

Theorem 3.1

A sequence $\{\alpha_n\}$ in $\Re$ has a unique limit.

Proof

Suppose $\{\alpha_n\}$ has two limits α_1 *and* α_2. That is $\lim_{n\to\infty}\alpha_n = \alpha_1$ and $\lim_{n\to\infty}\alpha_n = \alpha_2$. $\Rightarrow$ Given $\varepsilon > 0$, $\exists$ N_1 *and* N_2 such that

$$|\alpha_n - \alpha_1| < \frac{\varepsilon}{2},\ \forall n > N_1$$

and

$$|\alpha_n - \alpha_2| < \frac{\varepsilon}{2},\ \forall n > N_2.$$

By the triangle inequality $\forall n > N = \max\{N_1, N_2\}$,

$$|\alpha_1 - \alpha_2| = |\alpha_1 - \alpha_n + \alpha_n - \alpha_2| \le |\alpha_1 - \alpha_n| + |\alpha_n - \alpha_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$\Rightarrow |\alpha_1 - \alpha_2| < \varepsilon, \forall \varepsilon > 0.$$

$\Rightarrow |\alpha_1 - \alpha_2| = 0.$ Since $\varepsilon > 0$ is arbitrary $\Rightarrow \alpha_1 - \alpha_2 = 0 \Rightarrow \alpha_1 = \alpha_2.$ □

Theorem 3.2

Every convergent sequence of real numbers is bounded.

Proof

Suppose that α_n converges to α i.e. $\lim_{n\to\infty}\alpha_n = \alpha$ *and* let $\in = 1$ be given. Then $\exists$ a natural number $N = N(\in)$ such that

$$|\alpha_n - \alpha| < 1, \quad \forall\ n > N.$$

$$\Rightarrow |\alpha_n| = |\alpha_n - \alpha + \alpha| \le |\alpha_n - \alpha| + |\alpha| < 1 + |\alpha|, \quad \forall n > N$$

Let $M = \max\{\alpha_1, \alpha_2, ..., \alpha_{n-1}, 1 + |\alpha|\}$.

Then $|\alpha_n| \le M$, $\forall n$. Hence (α_n) is bounded. □

Conversely a bounded sequence need not be convergent for instance the sequence $\{\alpha_n\} = \left\{(-1)^n\left(1 + 1/n\right)\right\}$ is bounded but not convergent since $\{\alpha_n\} = \left\{-2, 3/2, -4/3, 5/4, -6/5, 7/6, ...\right\}$ diverges.

Theorem 3.3

Let $\{\alpha_n\}$ *and* $\{\beta_n\}$ be sequences of real numbers such that $\lim_{n\to\infty}\{\alpha_n\} = \alpha$ *and* $\lim_{n\to\infty}(\beta_n) = \beta$. Then

a) $\lim_{n\to\infty}\{\alpha_n + \beta_n\} = \alpha + \beta$

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b) $\lim_{n\to\infty} k\alpha_n = k\alpha, \quad k = \text{constant}$

c) $\lim_{n\to\infty}(\alpha_n\beta_n) = \alpha\beta$

d) $\lim_{n\to\infty}\left(\alpha_n / \beta_n\right) = \alpha / \beta$, provided that $\beta_n \neq 0$ for all n and $\beta \neq 0$.

Proof

(a) and (b) Exercise

c) Since $\lim_{n\to\infty}\{\alpha_n\} = \alpha$ *and* $\lim_{n\to\infty}\{\beta_n\} = \beta$ given $\in > 0$, $\exists M_1$ *and* M_2 such that $|\alpha_n| \leq M_1$ *and* $|\beta_n| \leq M_2, \ \forall n$

Now given $\varepsilon > 0$, $\exists$ N_1 *and* N_2 such that

$|\alpha_n - \alpha| \leq \dfrac{\in}{2M}$, when $n > N_1$ and

$|\beta_n - \beta| \leq \dfrac{\in}{2M}$, when $n > N_2$.

Now let $N = \max\{N_1, N_2\}$

$$\begin{aligned}
|\alpha_n\beta_n - \alpha\beta| &= |\alpha_n\beta_n - \alpha_n\beta + \alpha_n\beta - \alpha\beta| \\
&= |\alpha_n(\beta_n - \beta) + \beta(\alpha_n - \alpha)| \\
&\leq |\alpha_n|\,|\beta_n - \beta| + |\beta|\,|\alpha_n - \alpha| \\
&\leq M\,|\beta_n - \beta| + M\,|\alpha_n - \alpha| \\
&< \frac{\in}{2} + \frac{\varepsilon}{2} = \in, \quad \forall n > N.
\end{aligned}$$

$\Rightarrow \quad \alpha_n\beta_n \to \alpha\beta$ since $\in > 0$ was arbitrary.

(d) Since $(\alpha_n / \beta_n) = (\alpha_n)(1/\beta_n)$ it suffices from part (c) to show that $\lim_{n\to\infty}(1/\beta_n) = 1/\beta$.

That is give $\in > 0$ we must make

$$\left|\frac{1}{\beta_n} - \frac{1}{\beta}\right| = \left|\frac{\beta - \beta_n}{\beta_n\beta}\right| < \in, \text{ for all } n \text{ sufficiently large.}$$

Since $\beta \neq 0, \exists N,$ such that $n > N \Rightarrow$

$|\beta_n - \beta| < \dfrac{|\beta|}{2}$. Thus for $n > N$, we have

$$|\beta_n| = |\beta - (\beta - \beta_n)| \geq |\beta| - |\beta - \beta_n| > |\beta| - \frac{|\beta|}{2} = \frac{|\beta|}{2}$$

(Show that $|x - y| > |x| - |y|$ and $|x| - |y| \leq |x - y|$)

There also exists N_2 such that $n \geq N_2$ implies that $|\beta_n - \beta| < \dfrac{1}{2} \in |\beta|^2$

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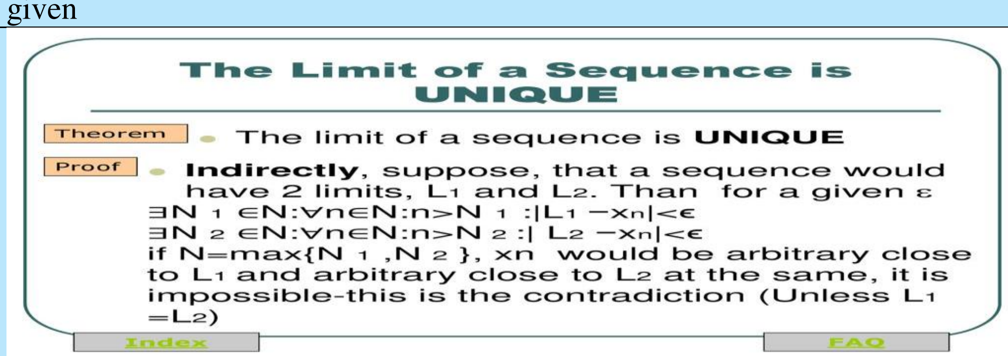
Let $N = \max\{N_1, N_2\}$. *Then* $n > N$ implies that

$$\left|\frac{1}{\beta_n} - \frac{1}{\beta}\right| = \left|\frac{\beta - \beta_n}{\beta_n \beta}\right| < \frac{2}{|\beta|^2} \cdot \frac{1}{2}|\beta|^2 = \varepsilon$$

Hence $\lim(1/\beta_n) = 1/\beta$.

E-tivity 3.2.2: Properties of limits

Numbering and pacing and sequencing	3.2.2
Title	Properties of limits
Purpose	To expose you to properties of limits
Brief summary of overall task	Watch the video on Uniqueness of limit of a sequence by The Grade Academy and Limit- Introduction by Zor Shekhtman and respond to the questions given
Spark	
Individual contribution	• Watch the two videos • Reflect on Theorems 3.1, 3.2 and 3.3
Interaction begins	• Post your answers on discussion forum 3.2.2 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take one hour.
Next	Subsequences and Cauchy sequences

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3.2.3 Subsequences and Cauchy Sequences

3.2.3.1 Subsequences

Definition 3.3 (Subsequence): Let X be a set and $\{\alpha_n\}$ be a sequence in X. Let (n_k) be a sequence of positive integers such that $n_1 < n_2 < \dots$. Then the sequence (α_{n_k}) is a subsequence of $\{\alpha_n\}$.

Example 3.7

If $\{\alpha_n\} = \{1, 1/2, 1/3, 1/4, 1/5, \dots\}$ then $\{\alpha_{n_k}\} = \{1/2, 1/4, 1/6, \dots\}$ is a subsequence of $\{\alpha_n\}$.

Theorem 3.4

A sequence $\{\alpha_n\}$ of real numbers converges to α if and only if every subsequence of $\{\alpha_n\}$ converges to α.

Proof

$\Rightarrow$ Suppose $\alpha_n \to \alpha$. We show that every subsequence (α_{n_k}) of $\{\alpha_n\}$ converges to α.

Thus for every $\varepsilon > 0$ there exists an integer N such that

$|\alpha_n - \alpha| < \in \alpha$, $\forall n > N$.

Take $n_k > N$, then

$|\alpha_{n_k} - \alpha| < \in \alpha$, $\forall n_k > N$.

Since (α_{n_k}) is any subsequence of $\{\alpha_n\}$, it follows that every subsequence of $\{\alpha_n\}$ converges to α whenever $\alpha_n \to \alpha$

$\Leftarrow$ Conversely, let every subsequence of $\{\alpha_n\}$ converge to α. Then $\alpha_n \to \alpha$ for $\{\alpha_n\}$ is a subsequence of itself. □

Example 3.8

Consider the sequence defined by

$$\alpha_n = \begin{cases} 1, & \text{if n is even} \\ 0, & \text{if n is odd} \end{cases}$$

That is $\{\alpha_n\} = \{0, 1, 0, 1, 0, \dots\}$. Then the subsequence

$\{\alpha_{2n}\} = \{1, 1, 1, \dots\}$ and converges to 1.

But the subsequence

$\{\alpha_{2n-1}\} = \{0, 0, 0, \dots\}$ and converges to 0.

Since $0 \neq 1$, then $\{\alpha_n\}$ diverges.

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3.2.3.2 Cauchy sequences

Definition 3.4: A sequence $\{\alpha_n\}$ of real numbers is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists an integer $\mathrm{N} = N(\varepsilon)$ such that

$|\alpha_m - \alpha_n| <\in \alpha$, $\forall\, m,n > N$.

Example 3.9

Show that the sequence $\left\{\frac{1}{2^n}\right\}$ is Cauchy.

Proof

We need to show that given $\varepsilon > 0$ there exists an integer $\mathrm{N} = N(\varepsilon)$ such that

$|\alpha_m - \alpha_n| <\in \alpha$, $\forall\, m,n > N$.

Let $m = n + p,\; p \in \mathrm{N}$, then

$$\Rightarrow |x_m - x_n| = \left|\frac{1}{2^{n+p}} - \frac{1}{2^n}\right| = \frac{1}{2^n}\left(1 - \frac{1}{2^p}\right) < \varepsilon .$$

Since $\left(1 - \frac{1}{2^p}\right) < 1$, then $\frac{1}{2^n} < \varepsilon$

$$\Rightarrow 2^n > \frac{1}{\varepsilon} \Rightarrow n > \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln 2}.$$

Choose $N = \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln 2}$, hence an integer N exists such that

$|\alpha_m - \alpha_n| <\in \alpha$, $\forall\, m,n > N$.

Hence $\left\{\frac{1}{2^n}\right\}$ is Cauchy.

Theorem 3.5

Every convergent sequence is Cauchy.

Proof

Suppose $\alpha_n \to \alpha$. Then for every $\varepsilon > 0$ there exists an integer N such that

$|\alpha_n - \alpha| <\in \frac{\varepsilon}{2}$, $\forall n > N$.

Take $m > n$, then we have

$|\alpha_m - \alpha| < \frac{\varepsilon}{2}$, $\forall\; m > n$

By the triangle inequality,

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$$\left|\alpha_m - \alpha_n\right| = \left|\alpha_m - \alpha + \alpha - \alpha_n\right| \le \left|\alpha_m - \alpha\right| + \left|\alpha - \alpha_n\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall m, n > N.$$

Thus $\{\alpha_n\}$ is Cauchy. □

Remark 3.1
Not every Cauchy sequence is convergent.

Example 3.10
Let $X = (0,1] \subset \Re$ and consider the sequence $\{\alpha_n\} = \left\{\frac{1}{n}\right\}$ in X. Then $\lim_{n\to\infty} \alpha_n = 0 \notin X$.

However, $\{\alpha_n\} = \left\{\frac{1}{n}\right\}$ is Cauchy. For let $\varepsilon > 0$ be given, then

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \varepsilon \Rightarrow n > \varepsilon.$$

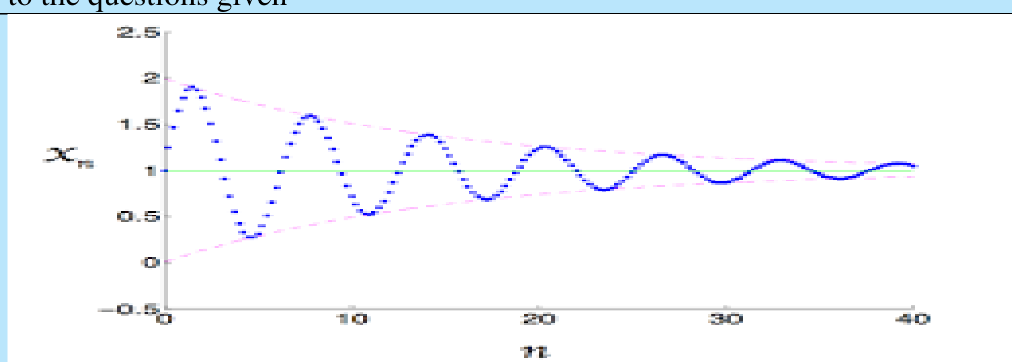
Hence taking $N = \frac{1}{\varepsilon}$, we have

$\left|\alpha_n\right| < \varepsilon, \forall\, n > N$.

Let $m > n$, then

$\left|\alpha_m - \alpha_n\right| <\in \alpha$, $\forall\, m, n > N$. Hence $\{\alpha_n\}$ is Cauchy.

E-tivity 3.2.3: Subsequences and Cauchy sequences

Numbering and pacing and sequencing	3.2.3
Title	Subsequences and Cauchy sequences
Purpose	To expose you to properties of subsequences and Cauchy sequences
Brief summary of overall task	Watch the video on Introduction to Cauchy sequence by Math Clinic respond to the questions given
Spark	
Individual contribution	• Watch the video on introduction to Cauchy sequence • Reflect on Theorems 3.4 and 3.5 • Prove that a sequence is convergent if and only if it is Cauchy

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	(Cauchy's Creterion of convergence)
Interaction begins	• Post your answers on discussion forum 3.2.3 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take one hour.
Next	Monotonic Sequences, Lim sup and Lim inf

3.2.4 Monotonic Sequences, Lim sup and Lim inf

Definition 3.5: Let $\{\alpha_n\}$ be a sequence of real numbers, then we say that $\{\alpha_n\}$

a) Is monotonic increasing, $\alpha_n \uparrow$ if $\alpha_n \le \alpha_{n+1}$, $\forall n \in N$.

b) Is monotonic decreasing, $\alpha_n \downarrow$ if $\alpha_n \ge \alpha_{n+1}$, $\forall n \in N$.

A sequence $\{\alpha_n\}$ is monotonic if it is increasing or decreasing.

Example 3.10

i) The sequence $\{1, 2, 4, 8, ...\} = \{2^n\}$ is monotonic increasing.

ii) The sequence $\left\{1, {}^{1}\!/_{3}, {}^{1}\!/_{9}, {}^{1}\!/_{27}, ...\right\} = \left\{{}^{1}\!/_{3^n}\right\}$ is monotonic decreasing.

Theorem 3.5

Let $\{\alpha_n\}$ be a monotonic sequence. Then $\{\alpha_n\}$ is convergent if and only if $\{\alpha_n\}$ is bounded.

Proof

Exercise

Definition 3.6 (Limit Superior, Limit Inferior): Let $\{\alpha_n\}$ be a bounded sequence. A subsequential limit of $\{\alpha_n\}$ is any real number that is the limit of some subsequence of

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$\{\alpha_n\}$. If S is the set of all subsequential limits of $\{\alpha_n\}$, then we define the limit superior of $\{\alpha_n\}$ to be

$Lim\sup\alpha_n = \overline{Lim\,\alpha_n} = \sup S$.

Similarly, we define the limit inferior of α_n to be

$Lim\inf\alpha_n = \underline{Lim\,\alpha_n} = \inf S$.

Example 3.11

Let $\alpha_n = (-1)^n + \frac{1}{n}$. We see that

$|\alpha_n| = \left|(-1)^n\right| + \frac{1}{|n|} \le 2$ for all *n*, so the sequence $\{\alpha_n\}$ is bounded. The first few terms are

$$0, 3/2, -2/3, 5/4, -4/5, 7/6, -6/7, \cdots.$$

The subsequence

$\{\alpha_{2n}\} = \{3/2, 5/4, 7/6, \cdots\}$ converges to 1 and the subsequence $\{\alpha_{2n-1}\} = \{0, -2/3, -4/5, -6/7, \cdots\}$ convergences to -1. Since these are the subsequential limits, we have

$$Lim\sup\alpha_n = \overline{Lim\,\alpha_n} = \sup\{-1, 1\} = 1$$

and

$$Lim\inf\alpha_n = \underline{Lim\,\alpha_n} = \inf\{-1, 1\} = -1.$$

Theorem 3.6

Let $\{\alpha_n\}$ be a sequence of real numbers, then $\{\alpha_n\}$ converges if and only if $\overline{Lim\,\alpha_n} = \underline{Lim\,\alpha_n}$ and the number is real.

Proof

We know that $\alpha_n \to \alpha$ iff every subsequence of $\{\alpha_n\}$ converges to α. Thus $S = \{\alpha\}$, where S is the set of subsequential limits of $\{\alpha_n\}$. Therefore $\sup S = \inf S = \alpha$, that is

$\overline{Lim\,\alpha_n} = \underline{Lim\,\alpha_n}$. Conversely let $\overline{Lim\,\alpha_n} = \underline{Lim\,\alpha_n}$, then $S = \{\alpha\}$. Therefore, every subsequence of $\{\alpha_n\}$ converges to α. But $\{\alpha_n\}$ is a subsequence of itself. Hence $\{\alpha_n\}$ converges to α. □

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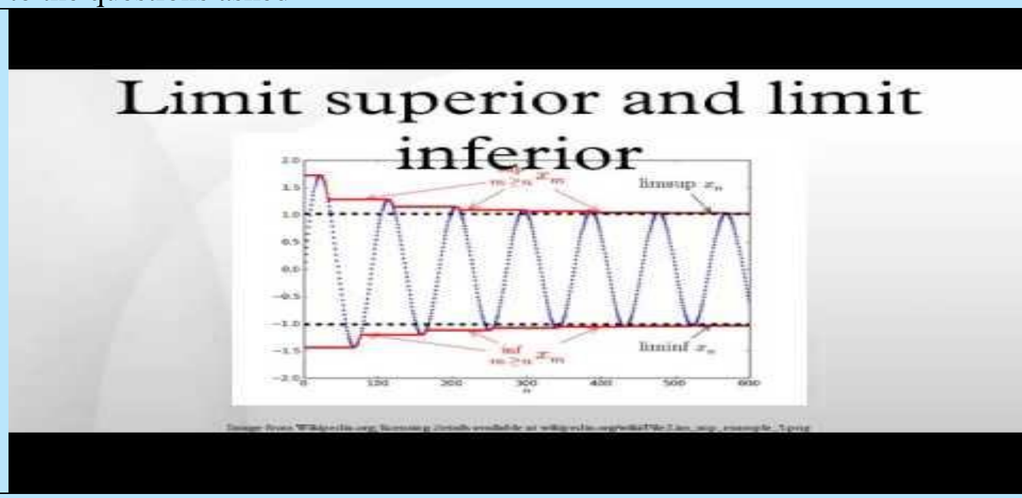
E-tivity 3.2.4: Monotonic Sequences, Lim sup and Lim inf

Numbering and pacing and sequencing	3.2.4
Title	Monotonic Sequences, Lim sup and Lim inf
Purpose	To enable you determine the limit superior and inferior of a sequence
Brief summary of overall task	Watch the video on Sequences more definitions by Linda Green and respond to the questions asked
Spark	
Individual contribution	• Watch the video on sequences more definitions • Find $\underline{Lim}\,\alpha_n$ and $\overline{Lim}\,\alpha_n$ given that $\{\alpha_n\} = \left\{ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, \frac{1}{4}, \frac{3}{4}, -\frac{3}{4}, \frac{1}{5}, \frac{4}{5}, -\frac{4}{5}, \dots \right\}$
Interaction begins	• Post your answers on discussion forum 3.2.4 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take one hour.
Next	Introduction to infinite series

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3.3 Assessment Questions

1. Show that

$$\lim_{n\to\infty}\left(\frac{4n^2-3}{5n^2-2n}\right)=\frac{4}{5}.$$

2. Prove that $x_n = \dfrac{n+1}{n+2}$ does **not** converge to 0.

3.4 References

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2. Bartle, R. G. (1976). *The elements of Real Analysis*. John Wiley and Sons, Inc. New York.
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3. Marsden, J. E. (1974). *Elementary Classical Analysis*. W. H. Freeman and Company, San Francisco.

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LECTURE FOUR

SERIES

4.1 Introduction

In this lesson we will study infinite series. We will consider several convergence test including Integral test, Comparison test, Alternating series test, Ratio test and Root test.

4.2 Learning Outcomes

By the end of this lesson the learner will be able to:

i) Discuss giving examples the concept of an infinite series
ii) Test convergence or divergence of series
iii) Investigate absolute and conditional convergence of a series

4.2.1 Introduction to Infinite Series

Definition 4.1 (Infinite Series): Let (a_n) be an infinite sequence of real numbers. The expression

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

is called an infinite series. The number a_n is called the nth term of the series. If we define

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s3_2 = a_1 + a_2 + a_3$$
.
.
.
$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

Then the sum s_n is called the nth partial sum of the series. The series $\sum_{n=1}^{\infty} a_n$ is said to be convergent if the sequence $\{s_n\}$ of partial sums converges and define

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to} s_n .$$

Otherwise the series is said to be divergent.

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Example 4.1
One of the most useful series is the geometric series

$$\sum_{n=0}^{\infty} ar^{n} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots,$$

where r is called the common ratio.

Now, the n[th] partial sum

$$s_n = \sum_{k=0}^{n} ar^k = a + ar + ar^2 + ... + ar^{n-1} = \frac{a\left(1-r^n\right)}{1-r}, \quad r \neq 1.$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$. Hence the geometric series converges to $\frac{a}{1-r}$ provided $|r| < 1$. If $r \geq 1$, the series diverges..

Example 4.2
The infinite series $\sum_{r=1}^{\infty} \frac{1}{n\left(n+1\right)}$ converges to 1.

Solution
We have the partial series given by

$$\begin{aligned} s_n &= \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + ... + \frac{1}{n\left(n+1\right)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + ... + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

Now $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[1 - \frac{1}{n+1}\right] = 1$.

Theorem 4.1
If $\sum a_n$ is a convergent series, then $\lim_{n\to\infty} a_n = 0$.

Proof
If $\sum a_n$ converges, them the sequence $\left(s_n\right)$ of partial sums have a finite limit, say a. But $a_n = s_n - s_{n-1}$, so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = a - a = 0$. □

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E-tivity 4.2.1: Introduction to infinite series

Numbering and pacing and sequencing	4.2.1
Title	Introduction to infinite series
Purpose	To expose you to the concept of convergence of infinite series
Brief summary of overall task	Watch the video on infinite series by Tayler Wallance and respond to the questions given
Spark	$\sum_{n=1}^{\infty}\left(\frac{2n!}{2^{2n}}\right)$
Individual contribution	• Watch the videos on infinite series • Determine whether the series $\sum_{n=1}^{\infty}\frac{n}{2n+5}$ converge or diverge.
Interaction begins	• Post your answers on discussion forum 4.2.1 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take one hour.
Next	Convergence Tests

4.2.2 Convergence Tests

4.2.2.1 The Integral Test:

Theorem 4.2 (Integral test)
Let f be a continuous function defined on $[1,\infty)$ and suppose that f is positive and decreasing; that is if $x_1 < x_2$ then $f(x_1) \geq f(x_2) > 0$
Then the series $\sum f(n)$ converges

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If $\lim_{n\to\infty}\left(\int_1^n f(x)\,dx\right)$ exists as a real number.

Example 4.3

Consider the series $\sum 1/n^p$ known as the p-series.

Let $f(x) = 1/x^p$ and recall that

For $p \neq 1$, $\int_1^n \frac{1}{x^p}\,dx = \frac{1}{1-p}\left(n^{1-p} - 1\right)$.

Then limit of $\int_1^n \frac{1}{x^p}\,dx$ as $n \to \infty$ will be finite if $p > 1$, and infinite if $p < 1$, thus by the integral test $\sum 1/n^p$ converges if $p > 1$ and diverges if $p < 1$.

When p=1 we get $\sum 1/n$, called the harmonic series which is divergent for

$$\int_1^n \frac{1}{x}\,dx = \lim_{n\to\infty}\left|\ln x\right|_1^n$$

$$\lim_{n\to\infty}\left|\ln n - \ln 1\right| = \infty .$$

Thus $\sum_{n=1}^{\infty} 1/n$ diverges.

Example 4.4

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges since it is a p-series with $p = 2 > 1$.

4.2.2.3 Comparison test

Theorem 4.3

Let $\sum a_n$ and $\sum b_n$ be infinite series of non-negative terms. That is, $a_n \geq 0$ and $b_n \geq 0$ for all *n*. Then

a) If $\sum a_n$ converges and $0 \leq bn \leq a_n$ for all *n*, then $\sum b_n$ converges.
b) If $\sum a_n > \infty$ *and* $0 \leq a_n \leq b_n$ for all *n*, then $\sum b_n > \infty$ (i.e. diverges).

Example 4.5

Consider the series

$\sum 1/(n+1)^2$. Now for all $n \in \mathrm{N}$

we have

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$$0 < \frac{1}{(n+1)^2} < \frac{1}{n(n+1)}$$

Since the series $\sum 1/_{n(n+1)}$ is convergent, so this implies that the series

$\sum \frac{1}{(n+1)^2}$ also converges.

Example 4.6
Determine whether each of the following series converges or diverges;

a) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

b) $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

c) $\sum_{n=0}^{\infty} \frac{1}{n!}$

Solution

i) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1} < \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty$,

i.e $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges being a p-series with $p = 3/2 > 1$.

Hence by comparison test $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges.

ii) $\frac{1}{2n-1} > \frac{1}{2n}$ for all $n \geq 1$.
Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} > \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} > \infty$$

i.e $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges being the harmonic series.

Hence $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges by comparison test.

iii) $\sum_{n=0}^{\infty} \frac{1}{n!} = 1+1+\frac{1}{2}+\frac{1}{6}+\cdots+\frac{1}{n!}+\cdots$

$< 1+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\cdots+\frac{1}{2^n}+\cdots$

$= 1+\sum_{n=0}^{\infty} \frac{1}{2^n} = 1+\left(\frac{1}{1-1/2}\right) = 3 < \infty$, geometric series with

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$a=1$ and $r=\frac{1}{2}$.

Hence $\sum_{n=0}^{\infty}\frac{1}{n!}$ converges by comparison test.

4.2.2.4 The Ratio and Root Test

Theorem 4.4 (Ratio Test)

Let $\sum a_n$ be a series of nonzero terms

a) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$, then the series converges absolutely.

b) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$, then the series diverges.

c) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$, then the test provides no information.

Example 4.7

Determines whether or not the series $\sum\frac{n^2}{2^n}$ converges or diverges.

Solution

$a_n=n^2/2^n$, $\quad a_{n+1}=\frac{(n+1)^2}{2^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{1}{2}\left(\frac{n+1}{n}\right)^2\right|=\lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right)^2=\frac{1}{2}<1.$$

Hence the series converges.

Example 4.8

Determine the convergence or divergence of the series $\sum\frac{1}{n^2}$.

Solution

$a_n=\frac{1}{n^2}$, $a_{n+1}=\frac{1}{(n+1)^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^2\right|=\lim_{n\to\infty}\left[\left(\frac{1}{1+\frac{1}{n}}\right)^2\right]=1.$$

The test provides no information.

Example 4.9

Determine the convergence or divergence of the series $\sum 3^n e^{-n}$.

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Solution

$a_n = 3^n e^{-n}, \quad a_{n+1} = 3^{n+1} e^{-(n+1)}$

$$\lim_{n\to\infty}\left|\frac{3^{n+1}}{e^{n+1}}\cdot\frac{e^n}{3^n}\right| = \lim_{n\to\infty}\frac{3}{e} = \frac{3}{e} > 1$$

Hence the series diverges.

Theorem 4.5 (Root Test)

Given a series $\sum a_n$ with all terms positive, and

let $\rho = \lim_{n\to\infty}|a_n|^{1/n}$

a) If $\rho < 1$ then the series converges absolutely.
b) If $\rho > 1$, then the series diverges.
c) If $\rho = 1$, then the test gives no information.

Example 4.10

The series $\sum 2^n e^{-n}$ converges

For $\lim_{n\to\infty}\left|\frac{2^n}{e^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{2}{e} = \frac{2}{e} < 1$.

Example 4.11

The series $\sum 3^n e^{-n}$ diverges for

$$\lim_{n\to\infty}\left|\frac{3^n}{e^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\left|\frac{3}{e}\right| = \frac{3}{e} > 1.$$

E-tivity 4.2.2: Tests of convergence

Numbering and pacing and sequencing	4.2.2
Title	Tests of convergence
Purpose	To expose you various tests of convergence of infinite series
Brief summary of overall task	Watch the videos on infinite geometric series by Brandon Gillins Integral test by UconnQCente, Convergence/Divergence of and harmonic and p-series by NCSSM Comparison test for series by Linda Green, Direct Comparison test by Sarah Peterson, Ratio test by Mathew Simmons, Ratio and Root tests by Tayler Wallance and respond to the questions asked

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Spark	Converges if P $>$ 1 and diverges if P $\leq$ 1 $$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots+\frac{1}{n^p}$$
Individual contribution	• Watch the seven videos • Determine whether the following series converge or diverge; a) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+1}}$ b) $\sum (-3)^n / n!$ c) $\sum \frac{(-5)^n}{2^n}$
Interaction begins	• Post your answers on discussion forum 4.2.2 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take 3 hours.
Next	Absolute Convergence and Conditional Convergence

4.2.3 Absolute Convergence and Conditional Convergence

4.2.3.1 Absolute convergence

Definition 4.2: Let (a_n) be a sequence in $\Re$. If the series $\sum |a_n|$ converges then the series $\sum a_n$ is to be absolutely convergent in $\Re$. If $\sum a_n$ converges but $\sum |an|$ diverges than $\sum a_n$ is said to be conditionally convergent.

Theorem 4.6
If a series is absolutely convergent then it is convergent.

Proof
Let $\sum \alpha_n$ be absolutely convergent. For each n

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$-|\alpha_n| \le \alpha_n \le |\alpha_n|$

So that

$0 \le \alpha_n + |\alpha_n| \le 2|\alpha_n|$

If $\sum|\alpha_n|$ converges them $\sum 2|\alpha_n|$ converges and by the comparison test, the non-negative series

$\sum(\alpha_n + |\alpha_n|)$ converges.

But $\alpha_n = (\alpha_n + |\alpha_n|) - |\alpha_n|$

Therefore $\sum \alpha_n$ converges being a differences of convergent series. □

Corollary 4.7

If $\sum \alpha_n$ diverges, then $\sum|\alpha_n|$ also diverges.

Proof

Exercise.

Example 4.12

The series $\sum(-1)^n 1/n^2$ converges absolutely for $\left|(-1)^n / n^2\right| = 1/n^2$

Thus $\sum\left|(-1)^n / n^2\right| = \sum 1/n^2$ which is converges being a p-series with p=2.

Hence the series $\sum (-1)^n / n^2$ converges.

4.2.3.2 The Alternating Series

Definition 4.3: .Alternating series is a series in which terms alternate between positive and negative values;

$$\sum_{n=1}^{\infty}(-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots.$$

Theorem 4.8 (Alternating Series Test/Leibniz's Theorem)

If (a_n) is a decreasing sequence of positive numbers and $\lim_{n\to\infty} a_n = 0$, then the series $\sum(-1)^{n+1} a_n$ converges.

Example 4.13

Since the sequence $\left(1/n\right)$ is decreasing and $\lim_{n\to\infty}\left(1/n\right) = 0$, we have that the alternating series $\sum(-1)^{n+1} 1/n$ converges.

Example 4.14

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The series $\sum_{n=2}^{\infty}\frac{(-1)^n}{n^n}$ converges since the sequence $\left(\frac{1}{n^n}\right)$ is decreasing and has $\lim_{n\to\infty}\left(1/n^n\right)=0$.

4.2.3.3 Conditional Convergence

Definition 4.4 (Conditional Convergence): A convergent sequence that is not absolutely convergent is said to be conditionally convergent. That is if $\sum a_n$ converges but $\sum|a_n|$ diverges, then $\sum a_n$ is said to be conditionally convergent.

Example 4.15

Determine the conditional convergence of $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$.

Solution

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}=1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots.$$

Since the sequence $\left(1/\sqrt{n}\right)$ is decreasing and $\lim_{n\to\infty}\left(1/\sqrt{n}\right)=0$, we have that the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ converges.

Now,

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{\sqrt{n}}\right|=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}>\infty$$

That is $\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ diverges being a p-series with $p=1/2<1$.

Hence $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ is conditionally convergent.

E-tivity 4.2.3: Absolute Convergence and Conditional Convergence

Numbering and pacing and sequencing	4.2.3
Title	Absolute Convergence and Conditional Convergence

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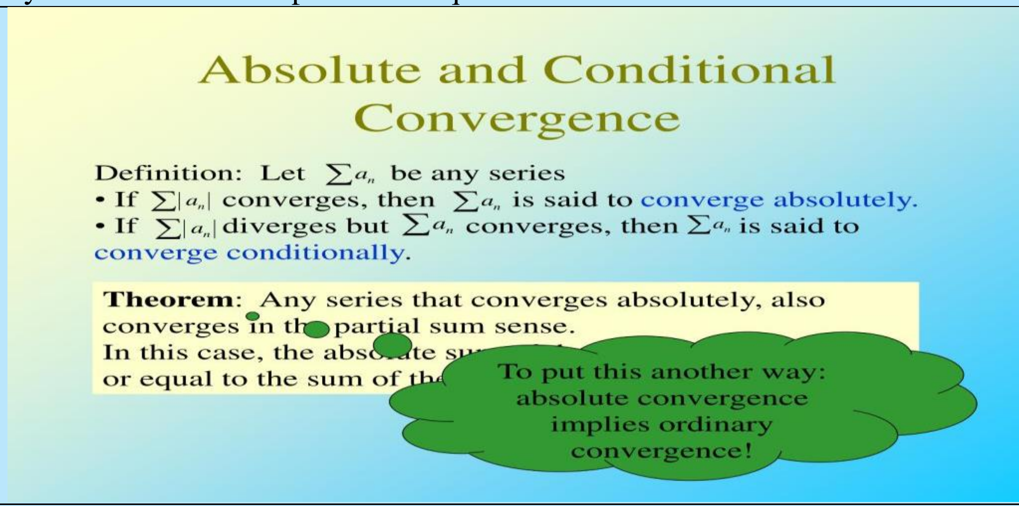
Purpose	To help you investigate absolute and conditional convergence of a given series
Brief summary of overall task	Watch the videos on [Absolute and conditional convergence]() by Andrew Misseldine, , [Alternating series]() by Tayler Wallance, [Conditional convergence]() by Jeff Suzuki and respond to the questions asked
Spark	
Individual contribution	• Watch the three videos • Determine whether the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^2-1}}$ converges absolutely, conditionally or not at all.
Interaction begins	• Post your answers on discussion forum 4.2.3 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take 2 hours.
Next	Countable sets

4.3 Assessment Questions

1. Apply the integral test to the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

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2. Determine the convergence or divergence of each of the following series. Give reasons in each case:

a) $\sum n^3 / 2^n$

b) $\sum_{n=1}^{\infty} \frac{2^n}{n^n}$

c) $\sum \frac{\sin^2 n}{n^2}$

d) $\sum \frac{(-1)^{n+1} n}{n^2 + 1}$

e) $\sum \frac{n!}{n^n}$

4.4 References

1. Chidume, C. E. (2006). *Foundations of Mathematical Analysis*. The Abdus Salam ICTP, Triese, Italy.

2. Bartle, R. G. (1976). *The elements of Real Analysis*. John Wiley and Sons, Inc. New York.
https://hayanihamudi.files.wordpress.com/2014/01/the-elements-of-real-analysis-by-robert-g-bartle.pdf

3. Marsden, J. E. (1974). *Elementary Classical Analysis*. W. H. Freeman and Company, San Francisco.

4. Rudin, W. (1966). *Principles of Mathematical Analysis*. McGraw-Hill, Inc. New York.

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LECTURE FIVE

COUNTABLE AND UNCOUNTABLE SETS

5.1 Introduction

In this lesson we will study countable and uncountable sets. We will show that the set of all integers Z is countable but the set of all real numbers $\Re$ is uncountable. We will also show that an infinite union of countable sets is a countable set.

5.2 Learning Outcomes

By the end of this lesson the learner will be able to:

i) Discuss giving examples properties of countable sets
ii) Discuss giving examples properties of uncountable sets

5.2.1 Countable sets

Definition 5.1 (Infinite set): A non-empty subset S of $\Re$ is said to be infinite if for any given natural number n, however large there is a subset of S containing more that n elements.

Example 5.1
The set $S = \{x \in \Re : x > 2\}$ is infinite.

Definition 5.2 (Countable set): A set S is said to be countable if it is finite or if it is infinite and its elements can be put into 1-1 correspondence with the set of natural numbers. That is S can be listed as infinite sequence $\{x_1, x_2, x_3, ...\}$

Example 5.2
The set of all even (odd) numbers is countable for its elements can be put into 1-1 correspondence with the set of natural numbers by definition. That is, there exists a function $f : N \to \text{E}$ defined by
$f(n) = 2n$ for even numbers and $f(n) = 2n - 1$ for odd numbers that sets up a 1-1 correspondence.

Example 5.3
The set Z of all integers is countable.

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Proof

Consider the following arrangement of the sets Z *and* N

$Z:0\ 1 -1\ 2 -2\ 3\ -3...$

$N:1\ 2\ 3\ 4\ \ 3\ \ 4\ \ 5...$

Then we can define a function $f:N \to Z$ which sets up a 1-1 correspondence as follows:

$$f(n) = \begin{cases} n/2, & if\ n\ is\ even \\ -\dfrac{n-1}{2}, & if\ n\ is\ odd \end{cases}.$$

5.2.1.1 Properties of countable sets

Theorem 5.1

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of countable sets. We show that $\bigcup_{n=1}^{\infty} A_n$ is countable.

Proof

For each A_n there exists 1-1 correspondence. $f_n : N \to A_n$, so we can write

$A_n = \{A_{n1}, A_{n2}, ...\}$ where $f_n(j) = A_{nj}$. We can now arrange the elements of $\bigcup_{n=1}^{\infty} A_n$ in a rectangular array:

$A_1 : A_{11} \to A_{12}\ A_{13} \to A_{14}....$

$A_2 : A_{21} \swarrow A_{22} \nearrow A_{23} \swarrow A_{24}....$

$A_3 : A \downarrow_{31} \nearrow A_{32} \swarrow A_{33}\ A_{34}...$

$A_4 : A_{41} \swarrow A_{42}\ A_{43}\ A_{44}...$

By moving along each diagonal of the array in the manner indicated, we obtain a listing of all elements in $\bigcup_{n=1}^{\infty} A_n$;

$\{A_{11}, A_{12}, A_{21}, A_{31}, A_{22}, A_{13}, A_{14}, A_{23}, A_{32}, A_{41}, ...\}$.

This procedure results in an infinite sequence of elements $\{z_1, z_2, z_3, ...\}$ of A which can be put into 1-1 correspondence with the set of natural numbers. Hence $\bigcup_{n=1}^{\infty} A_n$ is countable. □

E-tivity 5.2.1: Countable sets

Numbering and pacing and sequencing	5.2.1
Title	Countable sets
Purpose	To expose you to properties of countable sets

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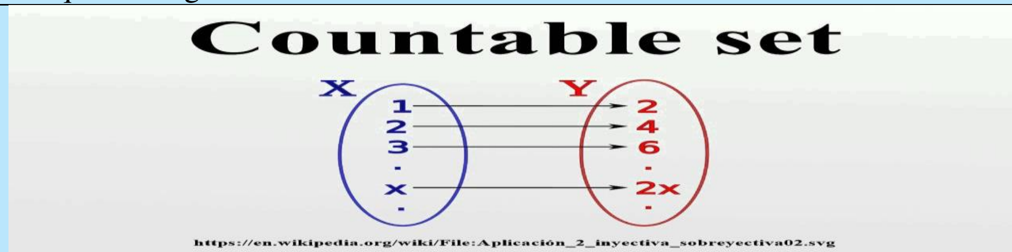
Brief summary of overall task	Watch the videos on Cardinality and countability by nptelhrd and respond to the questions given
Spark	
Individual contribution	• Watch the videos on Cardinality and countability • Answer the question; Show that the following sets are countable a) The set is all positive rational numbers b) The set of all negative rational numbers c) The set of all rational number
	• Post your answers on discussion forum 5.2.1 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take one hour
Next	Uncountable sets

5.2.2 Uncountable set

Definition 5.3: A set S is said to be uncountable if is not countable. That is if there is no one-one correspondence (function) between the elements of S and the set of natural numbers.

Example 5.4

The set $\Re$ of real numbers is uncountable.

Proof

Suppose that $\Re$ countable. Then $\Re$ can be written as a sequence of numbers $\Re = \{x_1, x_2, x_3 ...\}$. Choose $a_1, b_1 \in \Re$ such that $x_1 \notin [a_1, b_1]$. Having done this choose

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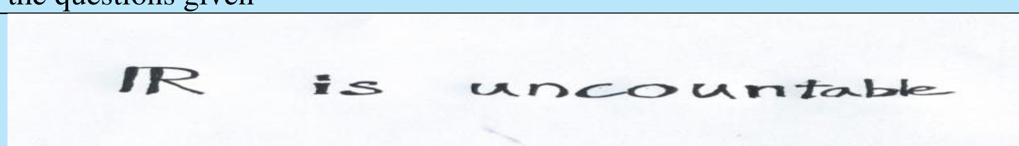
a_2, b_2 such that $a_1 \le a_2 < b_2 \le b_1$ and $x_2 \notin [a_2, b_2]$. Continuing this way, choose a_{n+1}, b_{n+1} such that $a_n \le a_{n+1} < b_{n+1} \le b_n$ and $x_{n+1} \notin [a_{n+1}, b_{n+1}]$.

Note that for each n, $x_n \notin [a_n, b_n]$ and the numbers $x_1\ x_2, .., x_n$ cannot be in the interval $[a_n, b_n]$.

Define $A = \{a_n | n \in Z^+\}$ and $\alpha = \sup A$. Clearly α is a real number and so $\alpha = x_N$ for some natural number N. Since α is an upper bound of A, $a_N \le \alpha$ and since b_N is an upper bound of A and α is the least upper bound of A, we see that $\alpha \le b_N$. We therefore conclude that $x_N \in [a_N, b_N]$, contradicting the choice of a_N and b_N which shows that the set $\Re$ is uncountable.

E-tivity 5.2.2: Uncountable sets

Numbering and pacing and sequencing	5.2.2
Title	Uncountable sets
Purpose	To expose you to the concept of uncountable sets.
Brief summary of overall task	Watch the video on and Reals are uncountable by Math Forge and respond to the questions given
Spark	
Individual contribution	• Watch the video on Reals are uncountable • Show that a) If the set F is countable and E is non-empty subset of F. Show that E is countable. b) If the set E is countable. Show that the set $E \cup \{a\}$ is countable
	• Post your answers on discussion forum 5.2.2 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion

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Schedule and time	This activity should take 1 hour
Next	Limits of functions

5.3 Assessment Questions

Show that the set $Q^{\subset}$ of irrational numbers uncountable

5.4 References

1. Chidume, C. E. (2006). *Foundations of Mathematical Analysis*. The Abdus Salam ICTP, Triese, Italy.

2. Bartle, R. G. (1976). *The elements of Real Analysis*. John Wiley and Sons, Inc. New York.
https://hayanihamudi.files.wordpress.com/2014/01/the-elements-of-real-analysis-by-robert-g-bartle.pdf

3. Marsden, J. E. (1974). *Elementary Classical Analysis*. W. H. Freeman and Company, San Francisco.

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LECTURE SIX

FUNCTIONS

6.1 Introduction

In this lesson we study limits and continuity of functions. We will prove limits by first principles and also consider both pointwise and uniform continuity of a function.

6.2 Learning Outcomes

By the end of this lesson the learner will be able to:

i) Prove limits of functions by first principles
ii) Determine whether a given function is pointwise continuous
iii) Determine whether a given function is uniformly continuous

6.2.1 Limits of functions

Consider the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

The function is defined for all *x* except at $x = 2$. If $x \neq 2$, *then* $f(x) = x + 2$.

Investigate what happens to the function $f(x)$ *as* $x \to 2$.

It can be shown that as $x \to 2 \quad f(x) \to 4$.

In this case 4 is called the limit value of $f(x)$ as $x \to 2$ written:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \left(\frac{x^2 - 4}{x - 2} \right) = 4.$$

Definition 6.1 (Limit): Let $S \subseteq \Re$ *and* $f : S \to \Re$ be a function. A real number L is said to be a limit point of *f* at point $a \in S$ if given any $\in > 0 \ \exists \ a \ \delta > 0$ such that if $x \in S$ $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

Note that *f* may or may not be defined as $x = a$.

We say that $f(x) \to L \ as \ x \to a$ and write

$\lim_{x \to a} f(x) = L$.

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Example 6.1
Let $f(x) = x^2 + 1 \quad \forall x \in \Re$. Prove that $\lim_{x \to 2} f(x) = 5$.

Solution
We need to show that given $\in > 0$, $\exists \quad a \quad \delta > 0$ such that

$$|x-2| < \delta \Rightarrow |f(x) - 5| < \in$$

Now for $x \in \Re$,

$$|f(x) - 5 = |x^2 + 1 - 5| = |x^2 - 4| = |x-2||x+2.|$$

Choose $\delta = \dfrac{\varepsilon}{|x+2|}$. Then whenever $|x-2| < \delta$ we have

$|f(x) - 5| = |x-2||x+2| < \delta|x+2| = \varepsilon.$ Hence. $\lim_{x \to 2} f(x) = 5$.

6.2.1.1 Properties of limits
(Proofs are similar to those as sequences)

a) If $\lim_{x \to a} f(x)$ exists then if is unique.

b) If $\lim_{x \to a} f(x) = L_1$ *and* $\lim_{x \to a} g(x) = L_2$, then

i) $\lim_{x \to a} \left[f(x) + g(x) \right] = L_1 + L_2$

ii) $\lim_{x \to a} \left[f(x) g(x) \right] = L_1 L_2$

iii) $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \dfrac{L_1}{L_2}$ provided $g(x) \neq 0 \ \forall x$ and $L_2 \neq 0$.

6.2.1.2 One-sided limits

Definition 6.2: Let $S \subseteq \Re$ and $f : S \to \Re$ be a function. If for every $x \in S$ $f(x) \to L \quad as \quad x \to a \ and \quad x > a$ always then we say that $x \to a$ from the right and write $x^+ \to a$ and say that

$$\lim_{x \to a^+} f(x) = L.$$

Similarly,
If $f(x) \to L \, as \, x \to a \, and \, x < a$ always then we say that $x \to a$ from the left and write $x \to a^-$ and say that

$$\lim_{x \to a^-} f(x) = L.$$

The limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ are called one-sided limits of f at *a*.

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Theorem 6.1
Let $S \subseteq \Re$. Then

$$\lim_{x \to a} f(x) = L \quad iff \quad \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

Example 6.2

a) $\lim_{x \to 0} \frac{x}{|x|}$ does not exist

For

$$\lim_{x \to 0^+} \frac{x}{|x|} = 1$$

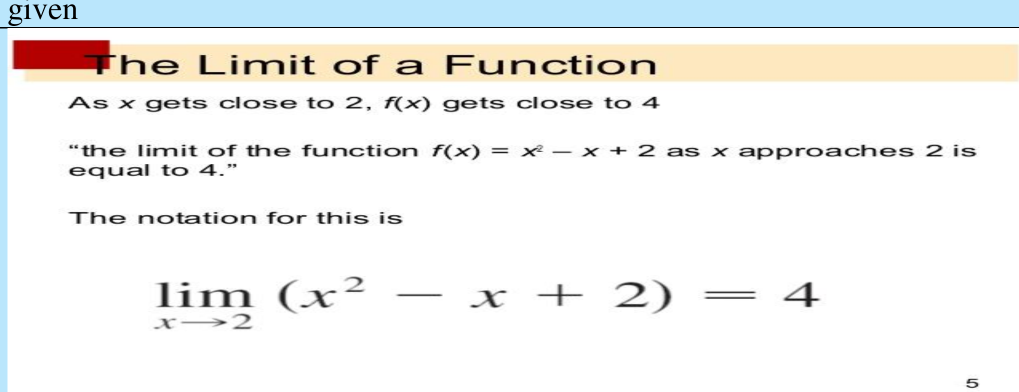
and

$$\lim_{x \to 0^-} \frac{x}{|x|} = -1$$

b) $\lim_{x \to 0} \frac{1}{x^2}$ and $\lim_{x \to 0} \frac{1}{x}$ have infinite limit.

c) $\lim_{x \to \infty} \frac{\sqrt{x - x}}{\sqrt{x + x}}$ has infinite limit.

E-tivity 6.2.1: Limits of functions

Numbering and pacing and sequencing	6.2.1
Title	Limit of functions
Purpose	To enable you prove limits by definition
Brief summary of overall task	Watch the videos on Epsilon delta definition of limit 03 by L Hub and and Epsilon delta definition of limit 06 by L Hub and respond to the questions given
Spark	
Individual contribution	• Watch the two videos • Show that from definition of a limit:

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	a) $\lim_{x\to 1}\frac{x^2-x+1}{x+1}=\tfrac{1}{2}$ b) $\lim_{x\to 2}\frac{x^3-8}{x^2-x-6}=\frac{12}{5}$ • Find a) $\lim_{x\to 0^+}\frac{4x}{\lvert x\rvert}$ c) $\lim_{x\to 0^-}\frac{4x}{\lvert x\rvert}$ b) $\lim_{x\to 1^-}\frac{x^2-1}{\lvert x-1\rvert}$
	• Post your answers on discussion forum 6.2.1 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take 1 hour
Next	Continuity of functions

6.2.2 Continuity of a function

Definition 6.3 (Continuity): A function $f:S\to\Re$ is said to be continuous at $x=a$ where $a\in S$ if

i) $\lim_{x\to a} f(x)$ exists finitely

ii) $\lim_{x\to a} f(x)=f(a)$.

Example 6.3

Let

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1} & ,x\neq 1\\ 2 & ,x=1\end{cases}$$

Then

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\frac{x^2-1}{x-1}=\frac{(x+1)(x-1)}{x-1}=2\,.$$

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Also

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2-1}{x-1} = x+1 = 2.$$

Thus, $\lim_{x\to 1} f(x) = 2 = f(1)$. Hence f is continuous $x=1$.

Definition 6.4 (Pointwise continuity): A function $f: S \to \Re$ is said to be continuous at $a \in S$ if given any $\in > 0$, $\exists \quad \delta > 0$ (depending both on $\in$ and the point) such that $|f(x) - f(a)| < \in$ whenever $|x-a| < \delta$.

Example 6.4

a) The function $f(x) = x^2$ is continuous at every $x = a$. For given by $\in > 0$, $a \ \ \delta > 0$ can be found such that

$|x-a| < \delta \Rightarrow |x^2 - a^2| < \in$.

Now,

$$|x^2 - a^2| = |(x-a)(x+a)|$$
$$= |x-a||x+a|.$$

Choose $\delta = \dfrac{\in}{|x+a|}$. Then whenever $|x-a| < \delta$ we have

$$|x^2 - a^2| = |x-a||x+a| < \delta|x+a| = \varepsilon.$$

Hence $f(x) = x^2$ is continuous at every $x = a$.

b) Prove that $f(x) = x^2 + 2x + 6$ or continuous at $x = 3$

Solution

Given $\in > 0$ we show that $\exists, \quad a \ \ \delta > 0$ such that $|x-3| < \delta$ implies

$|f(x) - f(3)| < \in$

Now,

$$|f(x) - f(3)| = |x^2 + 2x + 6 - 21| = |x^2 + 2x - 15| = |(x+5)(x-3)|$$
$$= |x+5||x-3|$$

Choose $\delta = \dfrac{\in}{|x+5|}$, then whenever $|x-3| < \delta$ we have

$$|f(x) - f(3)| < |x+5||x-3| < |x+5|\delta = \in.$$

Hence $f(x) = x^2 + 2x + 6$ or continuous at $x = 3$.

Definition 6.5 (Discontinuity of a function): Points where a function fails to be continuous are called points of discontinuity of the function. A function is said to be discontinuous at a point *x* if it is not continuous at that point.

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Example 6.5

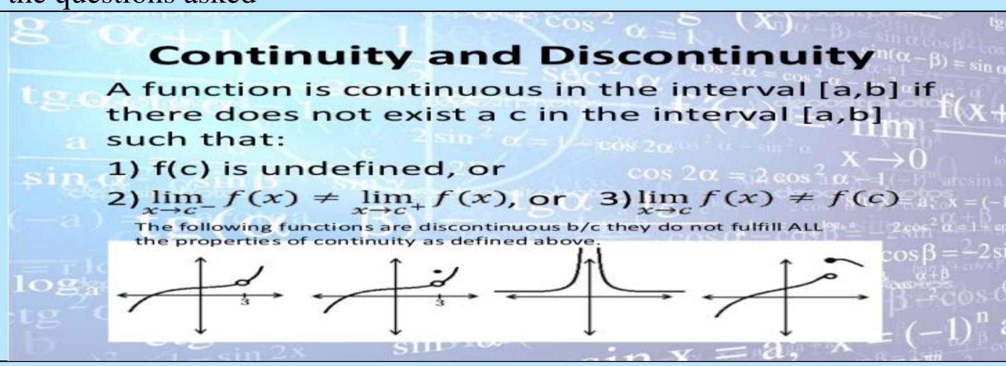
Let

$$f(x)=\begin{cases}\dfrac{x^2-a^2}{x-a} & ,x\neq a\\ 2 & ,x=a\end{cases}$$

Then $f(x)$ is discontinuous at $x=a$ since

$\lim_{x\to a^+} f(x)=2a=\lim_{x\to a^-} f(x)=2a\neq 2=f(a)$.

E-tivity 6.2.2: Continuity of functions

Numbering and pacing and sequencing	6.2.2
Title	Continuity of functions
Purpose	To expose to the concept of pointwise continuity of a function
Brief summary of overall task	Watch the video on functions limit continuity by Zor Shekhtman and answer the questions asked
Spark	
Individual contribution	• Watch the video on functions limit continuity • Show that $f(x)=x^2-5x+6$ is continuous at $x=3$. • Show that the function $f(x)=\begin{cases}\dfrac{2x^2-3x+1}{x-1}, & x\neq 1\\ 5 & ,x=1\end{cases}$ is discontinuous at $x=1$. How can we redefine $f(x)$ so that it is continuous at $x=1$.
	• Post your answers on discussion forum 6.2.2 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have

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	posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take 1 hour
Next	Uniform continuity

6.2.3 Uniform continuity

Recall (Pointwise continuity)

We said that $f: S \to \Re$ is said to be continuous at $x = a$ if for every $\varepsilon > 0$, $\exists \ \delta > 0$ such that $\left|f(x) - f(a)\right| < \in$ whenever $|x-a| < \delta$, where δ may depend on both $x \ and \in$.

That is for different points in S corresponding to pre-arranged $\in > 0$ we get different values of δ. If we can find a uniform δ that serves for all point of S corresponding to given $\in > 0$ then we say that f is uniformly continuous on S.

Example 6.6

Prove that the function $f(x) = 1/x$ is continuous at $x_0 \in (0, \infty)$.

Solution

Let $x_0 \in (0, \infty)$. Clearly $x_0 \neq 0$. Let $\in > 0$ be given. We want to find $a \ \delta > 0$ s.t $\forall \ x \in (0, \infty)$,

$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \in$.

Now, $\left|f(x) - f(x_0)\right| = \left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{|x||x_0|} < \frac{\delta}{|x||x_0|}$.

But $|x_0| - |x| \leq |x_0 - x| < \delta$. Hence,

$$\left|f(x) - f(x_0)\right| < \frac{\delta}{|x||x_0|}$$

$$< \frac{2\delta}{|x_0|} = \in \quad if \quad \delta = \frac{\varepsilon |x_0|}{2}$$

Hence, given $\in > 0$, choose $\delta = \min\left\{\frac{1}{2}|x_0|, \frac{\varepsilon |x_0|}{2}\right\}$

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Then $|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\in$ and so f is continuous at arbitrary $x\in(0,\infty)$ and δ depends on both x_0 *and* $\in$.

If this happens we say that f is not uniformly continuous on $(0,\infty)$.

Definition 6.6 (Uniform continuity): A function *f* defined on S is said to be uniformly continuous on S if for every $\in>0$, $\exists \; a \;\; \delta>0$ (depending on ε only) such that.

$$|f(x)-f(y)|<\in \text{ whenever } |x-y|<\delta \text{ for all } x,y\in S.$$

Example 6.7

Prove that $f(x)=2x$ is uniformly continuous on $\Re$.

Proof

Given $\in>0$ we want to make $|f(x)-f(y)|<\in$ by making *x* sufficiently close to *y*. That is for each $\varepsilon>0 \;\exists \; a \;\; \delta>0$ such that

$|x-y|<\delta \Rightarrow |f(x)-f(y)|<\in$

Now,

$|f(x)-f(y)|=|2x-2y|=2|x-y|<\in$. Choose $\delta=\frac{\varepsilon}{2}$ depending only on ε.

Then whenever $|x-y|<\delta$ we have

$|f(x)-f(y)|=2|x-y|<2\delta=\in$

Example 6.8

Prove that $f(x)=x^2$ is uniformly continuous on $[-5,5]$.

Solution

Now

$$\begin{aligned}|f(x)-f(y)|=|x^2-y^2|&=|(x-y)(x+y)|\\&\le|x-y||x+y|\\&\le 10|x-y|\end{aligned}$$

Choose $\delta=\frac{\varepsilon}{10}$ (depending on ε alone). Then whenever $|x-y|<\delta$ we have

$|f(x)-f(y)|\le 10|x-y|<10\delta=\varepsilon$.

Thus $f(x)=x^2$ is uniformly continuous on $[-5,5]$.

Example 6.9

The function $f(x)=\frac{1}{x}$ is uniformly continuous on $[2,\infty)$.

Proof

Given $\varepsilon>0$ and $x,y\in[2,\infty)$ we have,

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$$|f(x)-f(y)| = \left|\frac{1}{x}-\frac{1}{y}\right| = \frac{1}{xy}|x-y|$$
$$\leq \frac{1}{4}|x-y|$$

Choose $\delta = 4\varepsilon$, then whenever $|x-y| < \delta$ *we have* $|f(x)-f(y)| < \frac{1}{4}|x-y| < \frac{1}{4}\delta = \in$.

6.2.3.1 Negation of the definition of uniform continuity on S

If $\exists \in > 0$ such that $\forall \quad \delta > 0 \quad \exists \quad x, y \in S$ *such that* $|x-y| < \delta$ *but* $|f(x)-f(y)| \geq \in$.

Example 6.10

The function $f(x) = 1/x$ is not uniformly continuous in $(0,1)$.

Solution (BWOC)

Suppose that *f* is uniformly continuous in $(0,1)$ *so that* $\forall \in > 0 \quad \exists$ a $\delta > 0$ such that $|f(x)-f(y) < \in|$ whenever $|x-y| < \delta \quad \forall, x, y \quad \in (0,1)$

Let $x = \delta$ *and* $y = \frac{\delta}{1+\in}$, where $0 < \delta < 1$.

Then $|x-y| = \left|\delta - \frac{\delta}{1+\in}\right| = \left(\frac{\in}{1+\in}\right)\delta < \delta$.

But

$$|f(x)-f(y)| = \left|\frac{1}{x}-\frac{1}{y}\right| = \left|\frac{1}{\delta} - \frac{1+\varepsilon}{\delta}\right|$$
$$= \frac{\varepsilon}{\delta} > \in$$

Thus $|f(x)-f(y)| > \in$ which contradicts the fact that if is uniformly continuous in $(0,1)$.

Example 6.11

Show that $f(x) = x^2$ is not uniformly continuous on $\Re$.

Solution

Let $\in = 1$ and $\delta > 0$ be given. Let $x = \frac{1}{\delta}$ *and* $y = \frac{1}{\delta} + \frac{\delta}{2}$. Then $|x-y| = \frac{\delta}{2} < \delta$. But

$|f(x)-f(y)| = |x+y||x-y| = \left|\frac{1}{\delta}+\frac{1}{\delta}+\frac{\delta}{2}\right| \frac{\delta}{2} > \frac{2}{\delta}.\frac{\delta}{2} = 1$.

Thus f is not uniformly continuous on $\Re$. (why?) .

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Suppose $\in = 2 > 0$ would it as well work. We must show that given any $\varepsilon > 0\ \exists\ x, y \in \ \Re$ such that

$$|x-y| < \delta \Rightarrow \ |f(x)-f(y)| \geq 2$$

For *x*, if we let $y = x + \frac{\delta}{2}$

Then $|x-y| = \frac{\delta}{2} < \delta$.

To make

$$|f(x)-f(y)| = |x-y||x+y| < \varepsilon$$

We need to have $|x+y| \geq \frac{2}{8}$. This prompts us to choose $\delta = 1/8$ (How?).

Theorem 6.2

Let $f : S \rightarrow \Re$ be uniformly continuous on S and suppose that $\{x_n\}$ is a Cauchy sequence in S. Then $f(x_n)$ is a Cauchy sequence in $\Re$..

Proof

Given any $\in > 0$. Since f is uniformly continuous on $S\ \exists\ a\ \ \delta > 0$ such that $|f(x) - f(y)| < \in$ whenever $|x-y| < \delta \quad \forall\ x, y \in S$.

Since (x_n) is Cauchy $\exists$ a number N such that $|x_m - x_n| < \varepsilon$ whenever $m, n > N$.

Then for $m, n > N$ we have

$|f(x_n) - f(x)| < \in$ so $\{f(x_n)\}$ is a Cauchy sequence in $\Re$.

Theorem 6.3 (Uniform continuity)

Let $I = [a, b]$ be a closed bounded interval in $\Re$.. Let $f : I \rightarrow \Re$ be a continuous function on I. Then *f* is uniformly continuous on *I*. i.e. every continuous function on a closed bounded interval is uniformly continuous.

E.g. $f(x) = x^2$ is uniformly continuous on $[-5, 5]$.

E-tivity 6.2.3: Uniform continuity

Numbering and pacing and sequencing	6.2.3
Title	Uniform Continuity
Purpose	To expose to the concept of uniform continuity of a function
Brief summary of overall task	Watch the video on continuity and uniform continuity by IIT and respond to the questions given.

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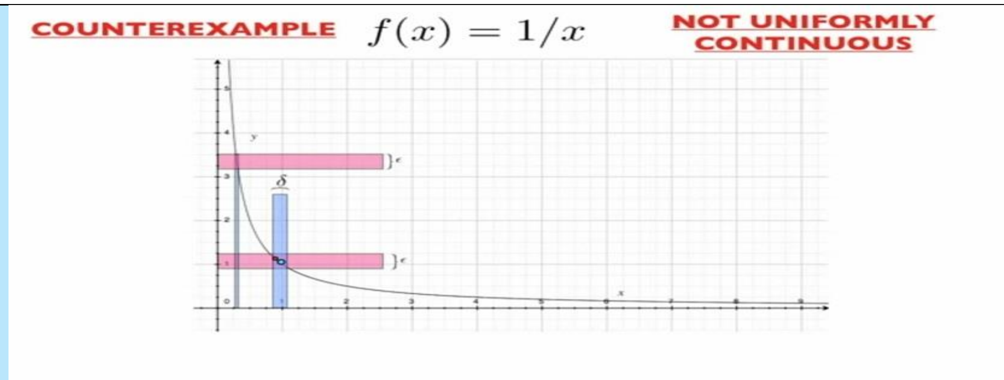
Spark	
Individual contribution	• Watch the videos on Continuity and uniform continuity • Prove that $f(x)=3x-5$ is uniformly continuous on $[-1,1]$.
	• Post your answers on discussion forum 6.2.3 • Read what your colleagues have posted. • In a sentence or two, comment on what two of your colleagues have posted keeping netiquette in mind
E-moderator interventions	• Focussing group discussion • Encouraging lurkers (quiet ones) to contribute • Providing feedback/ teaching points • Closing the discussion
Schedule and time	This activity should take 1 hour
Next	End of the lessons

6.3 Assessment Questions

1. Let $f(x)=\dfrac{x^2-4x-5}{x-5}$ for $x\neq 5.$ How should $f(x)$ defined so that it is continuous at?

2. Let $f(x)=x\sin\left(\frac{1}{x}\right)$ for $x\neq 0$ and $f(0)=0$ Show that $f(x)$ is continuous at $x=0.$

6.4 References

1. Chidume, C. E. (2006). *Foundations of Mathematical Analysis*. The Abdus Salam ICTP, Triese, Italy.

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2. Bartle, R. G. (1976). *The elements of Real Analysis*. John Wiley and Sons, Inc. New York.

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3. Marsden, J. E. (1974). *Elementary Classical Analysis*. W. H. Freeman and Company, San Francisco.

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ANSWERS

- ✓ **Lesson One**

Assessment Questions 1.3

Let $C = A + B,\ a = \sup A$, $b = \sup B$ and let $z \in C$ such that $z = x + y$, then $z \le a + b,\ \ \forall z \in C$. C is bounded above and therefore by completeness axiom C has a supremum. Let $c = \sup C$, then

$$c \le a + b \qquad \text{i)}$$

Let $\varepsilon > 0$ be given, then there exists $x \in A,\ \ y \in B$ such that $a - \varepsilon < x$ and $b - \varepsilon < y$.

Thus

$a + b - 2\varepsilon < x + y \le c$.

$\Rightarrow\ a + b \le c + 2\varepsilon$.

Letting $\varepsilon \to 0$ gives

$$a + b \le c \qquad \text{ii)}$$

From inequalities i) and ii), we obtain

$$c = a + b$$

That is

$\sup C = \sup A + \sup B.$

- ✓ **Lesson Two**

Assessment Questions 2.3

Since B is closed, then B^c is an open set. Therefore $A/B = A \cap B^c$ is open since it's a finite intersection of open sets.

- ✓ **Lesson Three**

Assessment Questions 3.3

1. We need to show that given any $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $\left| x_n - \frac{4}{5} \right| < \varepsilon, \quad \forall n > N$.

Now,

$$\left| x_n - \frac{4}{5} \right| = \left| \frac{4n^2 - 3}{5n^2 - 2n} - \frac{4}{5} \right| = \frac{8n - 15}{25n^2 - 10n} < \frac{8n}{25n^2 - 10n} = \frac{8}{25n - 10} < \varepsilon$$

$$\Rightarrow\ 25n\varepsilon - 10\varepsilon > 8$$

$$\Rightarrow\ 25n\varepsilon > 8 + 10\varepsilon$$

$$\Rightarrow n > \frac{8 + 10\varepsilon}{25}$$

Choose

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$N = N(\varepsilon) = \frac{8+10\varepsilon}{25}$, so that $\left|x_n - \frac{4}{5}\right| < \varepsilon, \quad \forall n > N.$

2. We must show that there exists an $\varepsilon > 0$ such that for all positive integers *N*, it's possible to have $n > N$ and $|x_n - 0| = |x_n| > \varepsilon$. Taking $\varepsilon = 1/2$ we see that,

   $|x_n - 0| = \frac{n+1}{n+2} = 1 - \frac{1}{n+2} > 1/2$, for any $n > N$ and $N \geq 1$.

   Hence x_n does **not** converge to 0.

- ✓ **Lesson Four**

Assessment Questions 4.3

1. Converges by Integral Test
2. a) Converges by Ratio Test
   b) Converges by Root Test
   c) Converges by Comparison Test
   d) Conditionally convergent
   e) Converges by Ratio Test

- ✓ **Lesson Five**

Assessment Questions 5.3

Suppose to the contrary that $\mathbb{Q}^c$ is countable. Then since $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ and $\mathbb{Q}$ is a countable set, then $\mathbb{R}$ is countable, a contradiction. Hence $\mathbb{Q}^c$ is uncountable.

- ✓ **Lesson Six**

Assessment Questions 6.3

1. $f(5) = 6$
2. We need to show that given any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(0)| < \varepsilon$ whenever $|x - 0| = |x| < \delta$.

   Now,

   $|f(x) - f(0)| = |x|\left|\sin\left(1/x\right)\right| \leq |x| < \delta$

   Choose $\delta = \varepsilon$, so that *f* is continuous at $x = 0$.

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RESOURCES

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